INTRODUCING SUPERSYMMETRIC FRIEZE PATTERNS AND LINEAR DIFFERENCE OPERATORS

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ABSTRACT. We introduce a supersymmetric analog of the classical Coxeter frieze patterns. Our approach is based on the relation with linear difference operators. We define supersymmetric analogs of linear difference operators called Hill's operators. The space of these "superfriezes" is an algebraic supervariety, which is isomorphic to the space of supersymmetric second order difference equations, called Hill's equations.

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INTRODUCTION

Frieze patterns were introduced by Coxeter [6], and studied by Coxeter and Conway [5]. Frieze patterns are closely related to classical notions of number theory, such as continued fractions, Farey series, as well as the Catalan numbers. Recently friezes have attracted much interest, mainly because of their deep relation to the theory of cluster algebras developed by Fomin and Zelevinsky; see [8]-[11]. This relation was pointed out in [9] and developed in [4]. Generalized frieze patterns were defined in [1]. Further relations to moduli spaces of configurations of points in projective spaces and linear difference equations were studied in [25, 26].

The main goal of this paper is to study superanalogs of Coxeter's frieze patterns. We believe that "superfriezes" introduced in this paper provide a first example of cluster superalgebra. We hope to investigate this notion in a more general setting elsewhere.

Our approach to friezes uses the connection with linear difference equations. The discrete Sturm-Liouville (one-dimensional Schrödinger) equation is a second order equation of the form:

$$V_i = a_i V_{i-1} - V_{i-2},$$

where the sequence (V_i) is unknown, and where the potential (or coefficient) (a_i) is a given sequence. Importance of linear difference equations is due to the fact that many classical sequences of numbers, orthogonal polynomials, special functions, etc. satisfy such equations. Linear difference equations with periodic coefficients, i.e., $a_{i+n} = a_i$, were recently used to study discrete integrable systems related to cluster algebras; see [28, 25, 26, 16]. It turns out that one particular case, where all the solutions of the above equation are antiperiodic:

$$V_{i+n} = -V_i$$

are of a special interest. We call this special class of discrete Sturm-Liouville equations Hill's equations. They form an algebraic variety which is isomorphic to the space of Coxeter's friezes.

We understand a frieze pattern as just another way to represent the corresponding Hill equation. Roughly speaking, a frieze is a way to write potential and solutions of a difference equation in the same infinite matrix. Friezes provide a very natural coordinate system of the space of Hill's equations that defines a structure of cluster manifold.

To the best of our knowledge, supersymmetric analogs of difference equations have never been studied. We introduce a class of supersymmetric difference equations that are analogous to Hill's (or Sturm-Liouville, one-dimensional Schrödinger) equations. We show that these difference equations can be identified with superfriezes. The main ingredient of difference equations we consider is the *shift operator* acting on sequences. In the classical case, the shift operator is the linear operator T defined by $(TV)_i = V_{i-1}$, discretizing the translation vector field $\frac{d}{dx}$. We define a supersymmetric version of T, as a linear operator $\mathfrak{T}$ acting on pairs of sequences and satisfying $\mathfrak{T}^2 = -T$. This operator is a discretization of the famous odd supersymmetric vector field $D = \partial_\xi - \xi\partial_x$. The corresponding "superfriezes" are constructed with the help of modified Coxeter's frieze rule where SL_2 is replaced with the supergroup $OSp(1|2)$.

Discrete Sturm-Liouville equations with periodic potential can be understood as discretization of the Virasoro algebra. Two different superanalogs of the Virasoro algebra are known as Neveu-Schwarz and Ramond algebras. The first one is defined on the supercircle $S^{1|1}$ related to the trivial 1-dimensional bundle over S^1, while the second one is associated with the twisted supercircle $S^{1|1}_+$ related to the Möbius bundle. The supersymmetric version we consider is the Möbius (or Ramond) one.

The main results of the paper are Theorems 2.6.2 and 2.7.2. The first theorem describes the main properties of superfriezes that are very similar to those of Coxeter's friezes. The second theorem identifies the spaces of superfriezes and Hill's equations.

The paper consists in three main sections.

In Section 1, we consider supersymmetric difference operators. The space of such operators with (anti)periodic solutions that we call Hill's operators is an algebraic supervariety.

In Section 2, we introduce analogs of Coxeter's friezes in the supercase. We establish the glide symmetry and periodicity of generic superfriezes. We prove that the space of superfriezes is an algebraic supervariety isomorphic to that of Hill's operators. We give a simple direct proof of the Laurent phenomenon occurring in superfriezes.

Each of these main section includes a short introduction outlining the main features of the respective classical theory.

Finally, in Section 3, we formulate and discuss some of the open problems.

The space of Coxeter's frieze patterns is a cluster variety associated to a Dynkin graph of type A; see [4]. Frieze patterns can be taken as the basic class of cluster algebras which explains the exchange relations and the mutation rules. We believe that the superfriezes that we introduce in this paper give rise to the notion of cluster superalgebras. This question will be treated in a consequent paper.

1. Supersymmetric linear difference operators

In this section, we introduce supersymmetric linear finite difference operators and the corresponding linear finite difference equations, generalizing the classical difference operators and difference equations.

The difference operators are defined using the supersymmetric shift operator.

1.1. **Classical discrete Sturm-Liouville and Hill's operators.** We start with a brief reminder of well-known second order operators.

The Sturm-Liouville operator (also known as discrete one-dimensional Schrödinger operators or Hill's operators) is a linear differential operator

$$\left(\frac{d}{dx}\right)^2 + u(x)$$

acting on functions in one variable.

The discrete version of the Sturm-Liouville operator is the following linear operator

$$L = T^2 - a\,T + \mathrm{Id},$$

acting on infinite sequences $V = (V_i)_{i \in \mathbb{Z}}$, where T is the *shift operator*

$$(TV)_i = V_{i-1},$$

and where $a = (a_i)$ is a given infinite sequence called the *coefficient*, or *potential* of the operator. The coefficient generates a diagonal operator, i.e., $(aV)_i = a_i V_i$. The sequence (a_i) is usually taken with values in $\mathbb{R}$, or $\mathbb{C}$.

Given an operator L, one can define the corresponding linear recurrence equation $L(V) = 0$, that reads:

(1.1) $$V_i = a_i V_{i-1} - V_{i-2},$$

for all $i \in \mathbb{Z}$. The sequence (V_i) is a *variable*, or *solution* of the equation.

Spectral theory of linear difference operators was extensively studied; see [16, 17] and references therein. The importance of second order operators and equations is due to the fact that many sequences of numbers and special functions satisfy such equations.

We will impose the following two conditions:

(a) the potential of the operator is n-periodic, i.e., $a_{i+n} = a_i$;

(b) *all* the solutions of the equation (1.1) are n-antiperiodic:

$$V_{i+n} = -V_i.$$

The condition (a) implies the existence of a *monodromy operator* $M \in \mathrm{SL}_2$ (defined up to conjugation). The space of solutions of the equation (1.1) is 2-dimensional; the monodromy operator is the restriction of the shift by the period, T^n, to the space of solutions. The condition (b) means that the monodromy operator is:

$$M = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that condition (b) implies condition (a), since the coefficients can be recovered from the solutions. A Sturm-Liouville operator satisfying conditions (a) and (b) will be called a *Hill's operator*.

The following statement is almost obvious; for a more general discussion see [26].

Proposition 1.1.1. *The space of Hill's operators is an algebraic variety of dimension $n - 3$.*

Indeed, the coefficients of the monodromy operator are polynomials in a_i's, and the condition $M = -\mathrm{Id}$ implies that the codimension is 3.

It turns out that the algebraic variety of Hill's operators has a geometric meaning. Consider the *moduli space* of configurations of n points in the projective line:

$$\{v_1, \ldots, v_n\} \subset \mathbb{P}^1, \qquad v_{i+1} \neq v_i$$

modulo the action of SL_2. This space will be denoted by $\widehat{\mathcal{M}}_{0,n}$. Note that this space is slightly bigger than the classical moduli space $\mathcal{M}_{0,n}$, which is the configuration space of n distinct points in $\mathbb{P}^1$. The following statement is a particular case of a theorem proved in [26].

Theorem 1.1.2. *If n is odd, then the algebraic variety of Hill's operators is isomorphic to $\widehat{\mathcal{M}}_{0,n}$.*

We will not give here a detailed proof of this statement; see [26]. The idea is as follows. Given a Hill operator, choose arbitrary basis of two linearly independent solutions, $V^{(1)}$ and $V^{(2)}$. One then defines a configuration of n points in $\mathbb{P}^1$ taking for every $i \in \mathbb{Z}$

$$v_i = (V_i^{(1)} : V_i^{(2)}).$$

This n-tuple of points is defined modulo linear-fractional transformations (homographies) corresponding to the choice of the basis of solutions.

1.2. **Supersymmetric shift operator.** Let $\mathcal{R} = \mathcal{R}_0 \oplus \mathcal{R}_1$ be an arbitrary supercommutative ring, and $\widehat{\mathcal{R}} = \mathcal{R} \oplus \xi\mathcal{R}$ its extension, where ξ is an odd variable.

We will be considering infinite sequences

$$V + \xi W := (V_i + \xi W_i), \qquad i \in \mathbb{Z},$$

where $V_i, W_i \in \mathcal{R}$. The above sequence is *homogeneous* if V_i and W_i are homogeneous elements of $\mathcal{R}$ with opposite parity.

Definition 1.2.1. The supersymmetric shift operator is the linear operator

(1.2) $$\mathfrak{T} = \frac{\partial}{\partial \xi} - \xi T,$$

where T is the usual shift operator. More explicitly, the action of $\mathfrak{T}$ on sequences is given by

$$\mathfrak{T}(V + \xi W)_i = W_i - \xi V_{i-1}.$$

Remark 1.2.2. The operator $\mathfrak{T}$ can be viewed as a discrete version (or "exponential") of the odd vector field

$$D = \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial x},$$

satisfying $D^2 = -\frac{\partial}{\partial x}$. This vector field is often called the "SUSY-structure", or the contact structure in dimension $1|1$; for more details see [20, 18, 19]. The vector field D is characterized as the unique odd left-invariant vector field on the abelian supergroup $\mathbb{R}^{1|1}$; see Appendix. We believe that the operator $\mathfrak{T}$ is a natural discrete analog of D because of the following properties (that can be checked directly):

(i) One has $\mathfrak{T}^2 = -T$.
(ii) The operator $\mathfrak{T}$ is equivariant with respect to the following action of $\mathbb{Z} \oplus \mathcal{R}_1$ on sequences in $\widehat{\mathcal{R}}$:

$$(k, \lambda) : (V + \xi W)_i \longmapsto V_{i+k} - \lambda W_{i+k} + \xi \left(\lambda V_{i+k-1} + W_{i+k} \right),$$

which is a discrete version of the (left) action of the supergroup $\mathbb{R}^{1|1}$ on itself, see Appendix.

It is natural to say that $\mathfrak{T}$ is a difference operator of *order* $\frac{1}{2}$.

1.3. Supersymmetric discrete Sturm-Liouville operators. We introduce a new notion of *supersymmetric discrete Sturm-Liouville operator* (or one-dimensional Schrödinger operator), and the corresponding recurrence equations.

Definition 1.3.1. The supersymmetric discrete Sturm-Liouville operator with potential U is the following odd linear difference operator of order $\frac{3}{2}$:

$$(1.3) \qquad\qquad \mathcal{L} = \mathfrak{T}^3 + U \mathfrak{T}^2 + \Pi,$$

where U is a given odd sequence:

$$U_i = \beta_i + \xi a_i,$$

with $a_i \in \mathcal{R}_0$, $\beta_i \in \mathcal{R}_1$, and where Π is the standard *parity inverting* operator:

$$\Pi \left(V + \xi W \right)_i = W_i + \xi V_i.$$

More explicitly, the operator $\mathcal{L}$ acts on sequences as follows

$$
\begin{aligned}
\mathcal{L} \left(V + \xi W \right)_i \;\; = \;\; & W_i - W_{i-1} - \beta_i V_{i-1} \\
& + \xi \left(V_i - a_i V_{i-1} + V_{i-2} + \beta_i W_{i-1} \right).
\end{aligned}
$$

The corresponding linear recurrence equation $\mathcal{L} \left(V + \xi W \right) = 0$ is the following system:

$$
\begin{cases}
V_i & = \;\; a_i V_{i-1} - V_{i-2} - \beta_i W_{i-1}, \\
W_i & = \;\; W_{i-1} + \beta_i V_{i-1},
\end{cases}
$$

for all $i \in \mathbb{Z}$. It can be written in the matrix form:

$$(1.4) \qquad
\begin{pmatrix} V_{i-1} \\ V_i \\ W_i \end{pmatrix} = A_i \begin{pmatrix} V_{i-2} \\ V_{i-1} \\ W_{i-1} \end{pmatrix}, \qquad \text{where} \qquad A_i = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & a_i & -\beta_i \\ \hline 0 & \beta_i & 1 \end{array} \right).
$$

This is a supersymmetric analog of the equation (1.1). It is easy to check that the matrix in the right-hand-side belongs to the supergroup $\mathrm{OSp}(1|2)$; see Appendix.

Remark 1.3.2. The continuous limit of the operator (1.3) is the well-known supersymmetric Sturm-Liouville Operator:

$$D^3 + U(x, \xi),$$

considered by many authors; see, e.g., [29, 13]. This differential operator is self-adjoint with respect to the Berezin integration. It is related to the coadjoint representation of the Neveu-Schwarz and Ramond superanalogs of the Virasoro algebra; see [27].

More precisely, $U(x, \xi) = U_1(x) + \xi U_0(x)$, and in the Neveu-Schwarz case the function is periodic: $U_0(x + 2\pi) = U_0(x)$ and $U_1(x + 2\pi) = U_1(x)$, while in the Ramond case it is (anti)periodic:

$$U_0(x + 2\pi) = U_0(x), \qquad U_1(x + 2\pi) = -U_1(x).$$

This corresponds to two different versions of the supercircle, the one related to the trivial bundle over S^1, and the second one related to the Möbius bundle.

1.4. **Supersymmetric Hill equations, monodromy and supervariety** $\mathcal{E}_n$. We will always assume the following periodicity condition on the coefficients of the Sturm-Liouville operator:

$$(1.5) \qquad\qquad a_{i+n} = a_i, \qquad \beta_{i+n} = -\beta_i.$$

Periodicity of coefficients of course does not imply periodicity or antiperiodicity of solutions. Any such equation has a *monodromy operator*, acting on the space of solutions

$$\begin{pmatrix} V_{i+n-1} \\ V_{i+n} \\ W_{i+n} \end{pmatrix} = M \begin{pmatrix} V_{i-1} \\ V_i \\ W_i \end{pmatrix}.$$

This operator can be represented as a matrix M_i which is a product of n consecutive matrices:

$$(1.6) \qquad\qquad M_i = A_{i+n-1} A_{i+n-2} \cdots A_{i+1} A_i,$$

where A_i is the matrix of the system (1.4), and therefore $M_i \in \mathrm{OSp}(1|2)$.

Definition 1.4.1. A *supersymmetric Hill equation* is an equation (1.4) such that all its solutions $(V + \xi W)$ satisfy the following (anti)periodicity condition:

$$(1.7) \qquad\qquad V_{i+n} = -V_i, \qquad W_{i+n} = W_i,$$

for all $i \in \mathbb{Z}$.

Since the space of solutions has dimension $2|1$, the condition (1.7) is equivalent to the fact that the monodromy matrix of such equation is:

$$(1.8) \qquad\qquad M_i = \left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array} \right).$$

Remarkably enough, the condition (1.8) does not depend on the choice of the initial i.

Lemma 1.4.2. *If the condition* (1.8) *holds for some i, then it holds for all $i \in \mathbb{Z}$.*

Proof. Let M_i be as in (1.8) for some i. By definition (1.6), and using (anti)periodicity of the coefficients (1.5), we have:

$$
M_{i+1} = A_{i+n} M_i A_i^{-1} = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & a_i & \beta_i \\ \hline 0 & -\beta_i & 1 \end{array} \right) \left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array} \right) \left(\begin{array}{cc|c} a_i & -1 & -\beta_i \\ 1 & 0 & 0 \\ \hline -\beta_i & 0 & 1 \end{array} \right)
$$

$$
= \left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array} \right).
$$

The result follows by induction. $\qquad\square$

The condition (1.7) is a strong condition on the potential of Hill's equation. More precisely, one has the following.

Proposition 1.4.3. *The space of Hill's equations satisfying the condition (1.7) is an algebraic supervariety of dimension $(n-3)\,|\,(n-2)$.*

Proof. The space of all Hill's equations with arbitrary (anti)periodic potential is just a vector space of dimensional $n|n$. The matrix M is given by the product (1.6), and therefore has polynomial coefficients in a's and β's. Thus, the condition (1.7) defines an algebraic variety. Furthermore, the condition (1.7) has codimension 3|2, i.e., the dimension of OSp(1|2). $\qquad\square$

We will denote by $\mathcal{E}_n$ the supervariety of Hill's equations satisfying the condition (1.7).

Remark 1.4.4. The condition (1.5) is manifestly a discrete version of the Ramond superalgebra, i.e., it corresponds to the Möbius supercircle. We do not know if the Neveu-Schwarz algebra can be discretize with the help of linear difference operators. This case would correspond to periodic sequences a's and b's, but then the monodromy M would have to be $\pm$Id. However, the case $M = $ Id cannot be related to Coxeter's friezes, and $M = -$Id is not an element of OSp(1|2).

1.5. Supervariety $\mathcal{E}_n$ for small values of n. Using the condition (1.8), one can write down explicitly the algebraic equations determining the supervariety $\mathcal{E}_n$. We omit straightforward but long computations.

a) The supervariety $\mathcal{E}_3$ has dimension 0|1. Every Hill's equation satisfying (1.7) for $n = 3$ has coefficients $a_i \equiv 1$, and $\beta_i = (-1)^i \beta$, where β is an arbitrary odd variable. This means that β_i satisfy the system:

$$
\left(\begin{array}{ccc} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{array} \right) \left(\begin{array}{c} -\beta_3 \\ \beta_1 \\ \beta_2 \end{array} \right) = 0.
$$

b) The supervariety $\mathcal{E}_4$ has dimension 1|2; the coefficients of the Hill's equation satisfy:

$$
\begin{aligned}
a_1 a_2 - 2 + \beta_1 \beta_2 &= 0, \\
a_2 a_3 - 2 + \beta_2 \beta_3 &= 0, \\
a_3 a_4 - 2 + \beta_3 \beta_4 &= 0, \\
a_4 a_1 - 2 + \beta_4 \beta_1 &= 0,
\end{aligned}
\quad \text{and} \quad
\left(\begin{array}{cccc} 0 & 1 & a_1 & 1 \\ -1 & 0 & 1 & a_2 \\ -a_1 & -1 & 0 & 1 \\ -1 & -a_2 & -1 & 0 \end{array} \right) \left(\begin{array}{c} -\beta_4 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{array} \right) = 0.
$$

The matrix of the linear system has rank 2.

c) One can check by a direct computation that the supervariety $\mathcal{E}_5$ is the $2|3$-dimensional supervariety defined by the following polynomial equations on 5 even and 5 odd variables $(a_1, \ldots, a_5, \beta_1, \ldots, \beta_5)$:

$$
\begin{aligned}
a_1a_2 - a_4 - 1 + \beta_1\beta_2 &= 0, \\
a_2a_3 - a_5 - 1 + \beta_2\beta_3 &= 0, \\
a_3a_4 - a_1 - 1 + \beta_3\beta_4 &= 0, \\
a_4a_5 - a_2 - 1 + \beta_4\beta_5 &= 0, \\
a_5a_1 - a_3 - 1 - \beta_5\beta_1 &= 0,
\end{aligned}
\qquad \text{and} \qquad
\begin{pmatrix}
0 & 1 & a_1 & a_4 & 1 \\
-1 & 0 & 1 & a_2 & a_5 \\
-a_1 & -1 & 0 & 1 & a_3 \\
-a_4 & -a_2 & -1 & 0 & 1 \\
-1 & -a_5 & -a_3 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
-\beta_5 \\
\beta_1 \\
\beta_2 \\
\beta_3 \\
\beta_4
\end{pmatrix}
= 0.
$$

Notice that exactly 3 even and 2 odd equations are independent.

d) The supervariety $\mathcal{E}_6$ is determined by six "even" equations on variables $(a_1, \ldots, a_6, \beta_1, \ldots, \beta_6)$, namely:

$$a_1 + a_3 + a_5 - a_3a_4a_5 - a_3\beta_4\beta_5 - a_5\beta_3\beta_4 - \beta_3\beta_5 = 0,$$

and its cyclic permutations, together with the following system of linear equations:

$$
\begin{pmatrix}
0 & 1 & a_1 & a_1a_2 - 1 & a_5 & 1 \\
-1 & 0 & 1 & a_2 & a_2a_3 - 1 & a_6 \\
-a_1 & -1 & 0 & 1 & a_3 & a_3a_4 - 1 \\
1 - a_1a_2 & -a_2 & -1 & 0 & 1 & a_4 \\
-a_5 & 1 - a_2a_3 & -a_3 & -1 & 0 & 1 \\
-1 & -a_6 & 1 - a_3a_4 & -a_4 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
-\beta_6 \\
\beta_1 \\
\beta_2 \\
\beta_3 \\
\beta_4 \\
\beta_5
\end{pmatrix}
= 0.
$$

Observe that the above equations for the even variables a_i are projected (modulo $\mathcal{R}_1$) to the equations defining classical Coxeter frieze patterns. The odd variables β_i in each of the above examples satisfy a systems of linear equations. The (skew-symmetric) matrices of the linear systems are nothing other than the matrices of Coxeter's friezes (for more details, see Section 2.1).

2. Superfriezes

In this section, we introduce the notion of *superfrieze*. It is analogous to that of Coxeter frieze, and the main properties of superfriezes are similar to those of Coxeter friezes. The space of all superfriezes is an algebraic supervariety isomorphic to the supervariety $\mathcal{E}_n$ of supersymmetric Hill's equations. In this sense, a frieze as just another, equivalent, way to record Hill's equations. Superfriezes provide a good parametrization of the space of Hill's equations.

2.1. Coxeter frieze patterns and Euler's continuants.
We start with an overview of the classical Coxeter frieze patterns, and explain an isomorphism between the spaces of Sturm-Liouville operators and that of frieze patterns. For more details, we refer to [6, 5, 1, 4, 24, 26, 30].

The notion of frieze pattern (or a frieze, for short) is due to Coxeter [6]. We define a frieze as an infinite array of numbers (or functions, polynomials, etc.):

$$
\begin{array}{ccccccccc}
\cdots & & 0 & & 0 & & 0 & & 0 & & \cdots \\
& \cdots & & 1 & & 1 & & 1 & & 1 & & \cdots \\
& & \cdots & & a_i & & a_{i+1} & & a_{i+2} & & a_{i+3} & & \cdots \\
& & & \cdots & & \cdots & & \cdots & & \cdots &
\end{array}
$$

where the entries of each next row are determined by the previous two rows via the following frieze rule: for each elementary "diamond"

$$
\begin{array}{ccc}
 & b & \\
a & & d \\
 & c &
\end{array}
$$

one has $ad - bc = 1$.

For instance, the entries in the next row of the above frieze are $a_i a_{i+1} - 1$, and the following row has the entries $a_i a_{i+1} a_{i+2} - a_i - a_{i+2}$.

Starting from generic values in the first row of the frieze, the frieze rule defines the next rows. For a *generic* frieze, the entries of the k-th row are equal to the following determinant

$$
K(a_i, \ldots, a_{i+k-1}) =
\begin{vmatrix}
a_i & 1 & & & \\
1 & a_{i+1} & 1 & & \\
 & \ddots & \ddots & 1 & \\
 & & 1 & a_{i+k-1}
\end{vmatrix}
$$

which is a classical *continuant*, already considered by Euler; see [22].

A frieze pattern is called *closed* if a row of 1's appears again, followed by a row of 0's:

$$
\begin{array}{ccccccc}
0 & & 0 & & 0 & & 0 & & \cdots \\
\cdots & 1 & & 1 & & 1 & & 1 & & \cdots \\
a_i & & a_{i+1} & & a_{i+2} & & a_{i+3} & & a_{i+4} \\
 & \cdots & & \cdots & & \cdots & & \cdots & \\
\cdots & 1 & & 1 & & 1 & & 1 & & \cdots \\
0 & & 0 & & 0 & & 0 & & \cdots
\end{array}
$$

The *width* m of a closed frieze pattern is the number of non-trivial rows between the rows of 1's. In other words, a frieze is closed of width m, if and only if $K(a_i, \ldots, a_{i+m}) = 1$, and $K(a_i, \ldots, a_{i+m+1}) = 0$, for all i.

Friezes introduced and studied by Coxeter [6] are exactly the closed friezes.

Let us recall the following results on friezes, [6, 5, 1]:

(1) A closed frieze pattern is horizontally periodic with period $n = m + 3$, that is, $a_{i+n} = a_i$.
(2) Furthermore, a closed frieze pattern has "glide symmetry" whose second iteration is the horizontal parallel translation distance n.
(3) A frieze pattern with the first row (a_i) is closed if and only if the Sturm-Liouville equation with potential (a_i) has antiperiodic solutions.

The name "frieze pattern" is due to the glide symmetry.

Proposition 2.1.1. *The space of closed friezes of width m is an algebraic variety of dimension m.*

Indeed, a closed frieze is periodic, so that one has a total of $2n = 2m + 6$ algebraic equations, $K(a_i, \ldots, a_{i+m}) = 1$ and $K(a_i, \ldots, a_{i+m+1}) = 0$ in n variables $a_1, \ldots, a_n$. It turns out that exactly 3 of these equations are algebraically independent, and imply the rest.

Based on results of [6], one can formulate the following statement: the two algebraic varieties below are isomorphic:

(1) the space of Sturm-Liouville equations with n-antiperiodic solutions;
(2) the space of Coxeter's frieze patterns of width $m = n - 3$;

for details see [25], [26]. The idea of the proof is based on the fact that every diagonal of a frieze pattern is a solution to the Sturm-Liouville equation with potential (a_i). More precisely, one has the recurrence formula for continuants

$$K(a_i, \ldots, a_j) = a_j K(a_i, \ldots, a_{j-1}) - K(a_i, \ldots, a_{j-2})$$

already known to Euler.

The above isomorphism allows one to identify Hill's equations and frieze patterns. The main interest in associating a frieze to a given Sturm-Liouville equation is that the frieze provides remarkable local coordinate systems. The coordinates are known as "cluster coordinates".

Example 2.1.2. A generic Coxeter frieze pattern of width 2 is as follows:

$$
\begin{array}{ccccccccc}
\cdots & & 1 & & 1 & & 1 & & \cdots \\
& x_1 & & \frac{x_2+1}{x_1} & & \frac{x_1+1}{x_2} & & x_2 & \\
\cdots & & x_2 & & \frac{x_1+x_2+1}{x_1 x_2} & & x_1 & & \cdots \\
& 1 & & 1 & & 1 & & 1 &
\end{array}
$$

for some $x_1, x_2 \neq 0$. (Note that we omitted the first and the last rows of 0's.) This example is related to the work of Gauss [12] on so-called *Pentagramma Mirificum*. It was noticed by Coxeter [6] that the values of various elements of self-dual spherical pentagons, calculated by Gauss, form a frieze of width 2.

2.2. **Introducing superfrieze.** Similarly to the case of classical Coxeter's friezes, a superfrieze is a horizontally-infinite array bounded by rows of 0's and 1's. Even and odd elements alternate and form "elementary diamonds"; there are twice more odd elements.

Definition 2.2.1. A superfrieze, or a supersymmetric frieze pattern, is the following array

$$
\begin{array}{ccccccccccc}
\cdots & & 0 & & & & 0 & & & & 0 \\
\cdots & 0 & & 0 & & 0 & & 0 & & 0 & \cdots \\
1 & & & & 1 & & & & 1 & & \cdots \\
& \varphi_{0,0} & & \varphi_{\frac{1}{2},\frac{1}{2}} & & \varphi_{1,1} & & \varphi_{\frac{3}{2},\frac{3}{2}} & & \varphi_{2,2} & \cdots \\
& & f_{0,0} & & & & f_{1,1} & & & & f_{2,2} \\
& \varphi_{-\frac{1}{2},\frac{1}{2}} & & \varphi_{0,1} & & \varphi_{\frac{1}{2},\frac{3}{2}} & & \varphi_{1,2} & & \varphi_{\frac{3}{2},\frac{5}{2}} & \cdots \\
f_{-1,0} & & & & f_{0,1} & & & & f_{1,2} & & \\
& \cdot\cdot\dot{} & & \cdot\cdot\dot{} & & \ddots & & \ddots & & \ddots & \ddots \\
& & f_{2-m,1} & & & & f_{0,m-1} & & & & f_{1,m} \\
\cdots & \varphi_{\frac{3}{2}-m,\frac{3}{2}} & & \varphi_{2-m,2} & & \cdots & & \varphi_{0,m} & & \varphi_{\frac{1}{2},m+\frac{1}{2}} & & \varphi_{1,m+1} \\
1 & & & & 1 & & & & 1 & & \\
\cdots & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
& & \cdots & 0 & & & & 0 & & & & 0 & \cdots
\end{array}
$$

where $f_{i,j}$ are even and $\varphi_{i,j}$ are odd, and where every *elementary diamond*:

$$
\begin{array}{ccc}
 & B & \\
\Xi & & \Psi \\
A & & D \\
\Phi & & \Sigma \\
 & C &
\end{array}
$$

satisfies the following conditions:

(2.1)
$$
\begin{aligned}
AD - BC &= 1 + \Sigma\Xi, \\
A\Sigma - C\Xi &= \Phi, \\
B\Sigma - D\Xi &= \Psi,
\end{aligned}
$$

that we call the *frieze rule*.

The integer m, i.e., the number of even rows between the rows of 1's is called the *width* of the superfrieze.

Remark 2.2.2. As usual, in the "supercase" there exists a projection to the classical case. Indeed, choosing all the odd variables $\varphi_{i,j} = 0$, the above definition is equivalent to the definition of a classical Coxeter frieze pattern with entries $f_{i,j}$.

Let us comment on the notation. The indices i, j of the entries of the frieze stand to number of diagonals of the frieze. More precisely, the first index i numbers South-East diagonals, and the second index j numbers North-East diagonals.

2.3. **More about the frieze rule.** The last two equations of (2.1) are equivalent to

$$
B\Phi - A\Psi = \Xi, \qquad D\Phi - C\Psi = \Sigma.
$$

Note also that these equations also imply $\Xi\Sigma = \Phi\Psi$, so that the first equation of (2.1) can also be written as follows:

$$
AD - BC = 1 + \Psi\Phi.
$$

Another way to express the last two equations of the frieze rule is to consider the odd entries neighboring to the elementary diamond. Then for every configuration

$$
\begin{array}{ccccc}
\widetilde{\Psi} & & & \widetilde{\widetilde{\Xi}} & \\
 & B & & & \\
\widetilde{\Phi} & \Xi & & \Psi & \widetilde{\Sigma} \\
 & A & & D & \\
 & \Phi & & \Sigma & \\
 & & C & &
\end{array}
$$

of the frieze, one has:

$$
B\left(\Phi - \widetilde{\Phi}\right) = A\left(\Psi - \widetilde{\Psi}\right), \qquad B\left(\Sigma - \widetilde{\Sigma}\right) = D\left(\Xi - \widetilde{\widetilde{\Xi}}\right).
$$

The relation to the group $\mathrm{OSp}(1|2)$ is as follows. One can associate an elementary diamond with every element of the supergroup $\mathrm{OSp}(1|2)$ (see Appendix) using the following formula:

$$
\begin{pmatrix}
a & b & \gamma \\
c & d & \delta \\
\hline
\alpha & \beta & e
\end{pmatrix}
\qquad \longleftrightarrow \qquad
\begin{array}{ccc}
 & -a & \\
\gamma & & \alpha \\
b & & -c \\
 & -\beta & \delta \\
 & d &
\end{array}
$$

so that the relations (2.1) coincide with the relations defining an element of $\mathrm{OSp}(1|2)$.

The frieze rule (2.1) implies the following elementary but useful properties.

Proposition 2.3.1. *(i) The entries $\varphi_{i,i}$ in the first non-trivial row of a generic superfrieze consist of pairs of equal ones: $\varphi_{i,i} = \varphi_{i+\frac{1}{2},i+\frac{1}{2}}$, where $i \in \mathbb{Z}$.*

(ii) The entries $\varphi_{i,i}$ in the last non-trivial row of a generic superfrieze consist of pairs of opposite ones: $\varphi_{i,i+m} = -\varphi_{i-\frac{1}{2},i+m-\frac{1}{2}}$, where $i \in \mathbb{Z}$.

2.4. **Examples of superfriezes.** The generic superfrieze of width $m = 1$ is of the following form:

$$
\begin{array}{ccccccccccccccc}
0 & & & & 0 & & & & 0 & & & & 0 & & \\
& 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
1 & & & & 1 & & 1 & & & & 1 & & & & 1 \\
& \xi & & \xi & & \xi' & & \xi' & & \xi - x\eta & & \xi - x\eta & & \eta & & \eta \\
& & x & & & & x' & & & & x & & & & x' \\
& \xi - x\eta & & x\eta - \xi & & \eta & & -\eta & & -\xi & & \xi & & -\xi' & & \xi' \\
1 & & & & 1 & & 1 & & & & 1 & & & & 1 \\
& 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
& & 0 & & & & 0 & & & & 0 & & & & 0
\end{array}
$$

where

$$
x' = \frac{2}{x} + \frac{\eta\xi}{x}, \qquad \xi' = \eta - \frac{2\xi}{x}.
$$

One can choose local coordinates (x, ξ, η) to parametrize the space of friezes.

The following example is the superanalog of the frieze from Example 2.1.2 related to the Gauss Pentagramma mirificum.

0		0		0		0		0		$\ldots$
	0	0	0	0	0	0	0	0	0	
$\ldots$	1		1		1		1		1	
	$-\zeta$	ξ	ξ	ξ'	ξ'	ν	ν	ζ^*	ζ^*	
y'		x		x'		x''		y		
	$-\eta'$	η^*	τ	η	τ'	η'	η^*	$-\tau$	η	
	x''		y		y'		x		x'	
	ν	$-\nu$	ζ^*	$-\zeta^*$	ζ	$-\zeta$	$-\xi$	ξ	$-\xi'$	
1		1		1		1		1		$\ldots$
	0	0	0	0	0	0	0	0	0	
$\ldots$	0		0		0		0		0	

The frieze is defined by the initial values (x, y, ξ, η, ζ), the next values are easily calculated using the frieze rule. The even entries of the superfrieze are as follows:

$$x' = \frac{1+y}{x} + \frac{\eta\xi}{x}, \qquad y' = \frac{1+x+y}{xy} + \frac{\eta\xi}{xy} + \frac{\zeta\eta}{y}, \qquad x'' = \frac{1+x}{y} + \frac{\eta\xi}{y} + \xi\zeta + \frac{x}{y}\zeta\eta.$$

For the odd entries of the superfrieze, one has:

$$\xi' = \eta - \frac{1+y}{x}\xi, \quad \eta' = \zeta - \frac{1+x+y}{xy}\xi - \frac{\xi\eta\zeta}{y}, \quad \tau' = \frac{1+y}{x}\zeta - \frac{1+x+y}{xy}\eta - \frac{\xi\eta\zeta}{x},$$

$$\zeta^* = \eta - y\zeta, \qquad \eta^* = \xi - x\zeta, \qquad \nu = \frac{(1+x)}{y}\eta - \xi - \zeta, \qquad \tau = x\eta - y\xi,$$

The superfriezes exhibited in the above example have many symmetries and periodicities. Our next task is to obtain these properties of superfriezes in general.

2.5. **Generic superfriezes and Hill's equations.** Like Coxeter's friezes, superfriezes enjoy remarkable properties, under some conditions of genericity. We begin with the most elementary way to define generic superfriezes.

Definition 2.5.1. A superfrieze is called generic if every even entry is invertible.

The following lemma explains the relation between superfriezes and linear difference equations.

Lemma 2.5.2. *The entries of every South-East diagonal of a generic superfrieze*

$$(W_i, V_i) := (\varphi_{j,i}, f_{j,i}),$$

where j is an arbitrary (fixed) integer, satisfy Hill's equation (1.4) with the potential $U_i = \beta_i + \xi a_i$, where a_i and β_i are given by the first two rows of the superfrieze, i.e., $a_i = f_{i,i}$ and $\beta_i = \varphi_{i,i}$.

Proof. We proceed by induction. Assume that the following fragment of a superfrieze:

$$
\begin{array}{ccccccccc}
 & & & & & 0 & & & \\
 & & & & \cdot^{\cdot^{\cdot}} & & \textcolor{red}{0} & & \\
 & & & B & & & & 1 & \\
 & & \textcolor{red}{\Xi} & & \textcolor{red}{\Psi} & & \cdot^{\cdot^{\cdot}} & & \textcolor{red}{\beta_i} \\
 & A & & & & D & & & a_i \\
 & & \textcolor{red}{\Phi} & & \textcolor{red}{\Sigma} & & \textcolor{red}{\Lambda} & & \cdot^{\cdot^{\cdot}} \\
 & & & C & & & & F & \\
 & & & & \textcolor{red}{\Omega} & & \cdot^{\cdot^{\cdot}} & & \\
 & & & & & E & & &
\end{array}
$$

satisfies the relations

$$F = a_i D - B - \beta_i \Psi, \qquad \Lambda = \Psi + \beta_i D$$

corresponding to the recurrence (1.4). We need to prove that these relations propagate on the next diagonal, i.e., that

$$E = a_i C - A - \beta_i \Phi, \qquad \Omega = \Phi + \beta_i C.$$

Indeed, using the superfrieze rule $D(\Omega - \Phi) = C(\Lambda - \Psi)$, we deduce $D(\Omega - \Phi) = \beta_i C D$, and canceling D, we obtain the second desired relation. For the even entries, we use the rule: $CF - DE = 1 + \Lambda\Omega$ together with $AD - BC = 1 - \Phi\Psi$. We have:

$$
\begin{aligned}
DE &= CF - 1 - \Lambda\Omega \\
&= a_i CD - CB - \beta_i C\Psi - 1 - \Lambda\Omega \\
&= a_i CD + 1 - \Phi\Psi - AD - \beta_i C\Psi - 1 - \Lambda\Omega \\
&= a_i CD - (\Phi + \beta_i C)\Psi - AD - (\Psi + \beta_i D)(\Phi + \beta_i C) \\
&= a_i CD - AD - \beta_i D\Phi,
\end{aligned}
$$

and again canceling D we obtain the desired relation.

Note that canceling D twice is allowed due the genericity assumption. $\qquad\square$

A similar property holds for North-East diagonals.

Lemma 2.5.3. *The entries of every North-East diagonal of a generic superfrieze*

$$(W_i^*, V_i^*) := (\varphi_{i+\frac{3}{2}, j+\frac{1}{2}}, \; f_{i+2, j}),$$

where j is an arbitrary (fixed) integer, satisfy the following Hill equation

$$
(2.2) \qquad
\begin{pmatrix} V_{i-1}^* \\ V_i^* \\ W_i^* \end{pmatrix}
=
\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & a_i & \beta_i \\ \hline 0 & -\beta_i & 1 \end{array} \right)
\begin{pmatrix} V_{i-2}^* \\ V_{i-1}^* \\ W_{i-1}^* \end{pmatrix},
$$

where $a_i = f_{i,i}$ and $\beta_i = \varphi_{i+\frac{1}{2}, i+\frac{1}{2}}$.

Proof. Consider the jth North-East diagonal $(V_i', W_i') := (f_{i,j}, \varphi_{i+\frac{1}{2}, j+\frac{1}{2}})$. As in the proof of Lemma 2.5.2, one establishes by induction the following system:

$$
\begin{aligned}
V_i' &= a_i V_{i+1}' - V_{i+2}' + \beta_i W_{i+1}', \\
W_i' &= \beta_i V_{i+1}' + W_{i+1}'.
\end{aligned}
$$

Inverting the matrix of the system and shifting the indices, one obtains (2.2). $\qquad\square$

Note that the difference between the equation (1.4) and the equation (2.2) is in the sign of the odd coefficients β_i.

The following properties are crucial for the notion of variety of friezes introduced in the sequel.

Proposition 2.5.4. *(i) A generic superfrieze is completely determined by the first two non-trivial rows, $\varphi_{i,i}$ and $f_{i,i}$, below the row of 1's.*

(ii) The entries $f_{j,i}$, $\varphi_{j,i}$ and $\varphi_{j-\frac{1}{2},i+\frac{1}{2}}$ of a generic superfrieze are polynomials in the entries β_i and a_i of the first two rows, defined by the recurrent formula:

$$(2.3) \qquad \begin{pmatrix} f_{j,i-1} \\ f_{j,i} \\ \varphi_{j,i} \end{pmatrix} = A_i \begin{pmatrix} f_{j,i-2} \\ f_{j,i-1} \\ \varphi_{j,i-1} \end{pmatrix},$$

where A_i is the matrix of the system (1.4), starting from the initial conditions

$$(2.4) \qquad\qquad (f_{j,j-3}, f_{j,j-2}, \varphi_{j,j-2}) = (-1, 0, 0),$$

and $\varphi_{j-\frac{1}{2},i+\frac{1}{2}} = f_{j,i}\varphi_{j-1,i} - f_{j-1,i}\varphi_{j,i}$.

Proof. Lemma 2.5.2 implies that every diagonal of a generic superfrieze is determined by β_i and a_i via the Hill equation (1.4). Therefore, the entries of the frieze are obtained as solutions $(V_i + \xi W_i)$ with the initial conditions

$$V_{-1} = 0, \quad V_0 = 1, \qquad W_0 = 0.$$

Finally, these initial conditions imply $V_{-2} = -1, W_{-1} = 0$. Hence the result. $\qquad\square$

Example 2.5.5. A generic superfrieze starts as follows:

$$\cdots \qquad\qquad 0 \qquad\qquad \cdots$$

$$0 \qquad\qquad\qquad\qquad 0$$

$$1 \qquad\qquad\qquad\qquad\qquad 1$$

$$\beta_0 \qquad\quad \beta_1 \qquad\qquad\quad \beta_1 \qquad\quad \beta_2$$

$$a_0 \qquad\qquad\qquad\qquad a_1 \qquad\qquad\qquad\qquad a_2$$

$$\begin{array}{c} a_0\beta_1 \\ +\beta_0 \end{array} \qquad \begin{array}{c} a_1\beta_0 \\ +\beta_1 \end{array} \qquad\qquad \begin{array}{c} a_1\beta_2 \\ +\beta_1 \end{array} \qquad \begin{array}{c} a_2\beta_1 \\ +\beta_2 \end{array}$$

$$\begin{array}{c} a_0a_1 - 1 \\ +\beta_0\beta_1 \end{array} \qquad\qquad\qquad\qquad \begin{array}{c} a_1a_2 - 1 \\ +\beta_1\beta_2 \end{array}$$

$$\begin{array}{c} \beta_0\beta_1\beta_2 \\ +\beta_0 - \beta_2 \\ +a_0a_1\beta_2 \\ +a_0\beta_1 \end{array} \qquad\qquad \begin{array}{c} \beta_0\beta_1\beta_2 \\ -\beta_0 + \beta_2 \\ +a_1a_2\beta_0 \\ +a_2\beta_1 \end{array}$$

$$\vdots \qquad\qquad \begin{array}{c} a_0a_1a_2 - a_0 \\ +\beta_0\beta_2 - a_2 \\ +a_0\beta_1\beta_2 \\ +a_2\beta_0\beta_1 \end{array} \qquad\qquad \vdots$$

$$\vdots$$

that can be deduced directly from (2.1).

The fact that the frieze is closed, i.e., ends with the rows of 1's and 0's, imposes strong conditions on the values of (β_i) and (a_j). These conditions will be described in Section 2.6.

2.6. **The glide symmetry and periodicity.** The properties of periodicity and glide symmetry are analogous to Coxeter's glide symmetry of frieze patterns. In the classical case, it was proved by Coxeter [6]. This periodicity is usually considered in contemporary works as an illustration of Zamolodchikov's periodicity conjecture; see [15] and references therein.

Lemma 2.6.1. *The entries of a superfrieze of width m satisfy the following periodicity property:*

$$\varphi_{i+n,j+n} = -\varphi_{i,j}, \qquad f_{i+n,j+n} = f_{i,j},$$

where $n = m + 3$; in particular, the entries of the first two rows satisfy $a_{i+n} = a_i$, $\beta_{i+n} = \beta_i$.

Proof. Let us first prove that the entries of the first two non-trivial rows $a_i = f_{i,i}$ and $\beta_i = \varphi_{i,i}$ are n-(anti)periodic. Indeed, consider the bottom part of the frieze:

$$f_{i-m-1,i-2}$$

$$\varphi_{i-m-\frac{3}{2},i-\frac{3}{2}} \qquad\qquad \varphi_{i-m-1,i-1}$$

$$f_{i-m-2,i-2} \qquad\qquad\qquad\qquad f_{i-m-1,i-1}$$

$$\varphi_{i-m-\frac{5}{2},i-\frac{3}{2}} \qquad\qquad 0 \qquad\qquad 0 \qquad\qquad \varphi_{i-m-1,i}$$

$$f_{i-m-3,i-2} \qquad\qquad\qquad\qquad 0 \qquad\qquad\qquad\qquad f_{i-m-1,i}$$

We use the odd "South-East relation" of Lemma 2.5.2 with $j = i - m - 1$:

$$\underbrace{\varphi_{i-m-1,i}}_{=0} = \varphi_{i-m-1,i-1} + \beta_i \underbrace{f_{i-m-1,i-1}}_{=1}$$

to obtain $\varphi_{i-m-1,i-1} = -\beta_i$. We use the odd "North-East relation" of Lemma 2.5.3 with $j = i - 2$:

$$\varphi_{i-m-\frac{3}{2},i-\frac{3}{2}} = -\beta_{i-m-3} \underbrace{f_{i-m-2,i-2}}_{=1} + \underbrace{\varphi_{i-m-\frac{5}{2},i-\frac{3}{2}}}_{=0}$$

to obtain $\varphi_{i-m-\frac{3}{2},i-\frac{3}{2}} = -\beta_{i-m-3}$. By Proposition 2.3.1, Part (ii), one deduces the antiperiodicity of the odd coefficients $\beta_i = -\beta_{i-m-3}$. Similarly, using the even relations, one deduces the periodicity of the even coefficients $a_i = a_{i-m-3}$.

Since the first two non-trivial rows determine the frieze, see Proposition 2.5.4, Part (i), the periodicity follows. $\qquad\square$

Furthermore, the following statement is analogous to the glide symmetry of friezes discovered by Coxeter [6].

Theorem 2.6.2. *A generic superfrieze satisfies the following glide symmetry*

$$
\begin{aligned}
f_{i,j} &= f_{j-m-1,i-2}, \\
\varphi_{i,j} &= \varphi_{j-m-\frac{3}{2},i-\frac{3}{2}}, \\
\varphi_{i+\frac{1}{2},j+\frac{1}{2}} &= -\varphi_{j-m-1,i-1},
\end{aligned}
$$
(2.5)

for all $i, j \in \mathbb{Z}$.

Proof. This statement readily follows from Lemmas 2.5.2, 2.5.3 and 2.6.1.

Indeed, choosing $j = 1$, the South-East diagonal $(W_i, V_i) = (\varphi_{1,i}, f_{1,i})$ is determined by the recurrence (1.4) and the initial condition

$$V_{-1} = 0, \quad V_0 = 1, \qquad W_0 = 0.$$

On the other hand, choosing $j = m+2$, the North-East diagonal $(W_i^*, V_i^*) = (\varphi_{i+\frac{3}{2}, m+\frac{5}{2}}, f_{i+2,m+2})$, is determined by the recurrence (2.2) and the same initial condition

$$V_{-1}^* = 0, \quad V_0^* = 1, \qquad W_0^* = 0.$$

The two recurrence relations differ only by the sign of the odd coefficients, therefore one has $(W_i, V_i) = (-W_i^*, V_i^*)$. The arguments for any j are similar, and we obtain

$$(\varphi_{j,i}, f_{j,i}) = (-\varphi_{i+\frac{3}{2}, m+j+\frac{3}{2}}, f_{i+2,m+j+1}).$$

Finally, using the antiperiodicty of the whole pattern established in Lemma 2.6.1, we deduce the set of relations (2.5). □

The above statement can be illustrated by the following diagram representing the diagonals of the superfrieze:

2.7. The algebraic variety of superfriezes: isomorphism with $\mathcal{E}_m$.
The above properties of generic superfriezes motivate the following important definition of the space of superfriezes that includes generic ones.

Definition 2.7.1. The algebraic supervariety of superfriezes is the supervariety defined by $2n$ even and n odd polynomial equations on variables $(a_1, \ldots, a_n, \beta_1, \ldots \beta_n)$ expressing that the last three rows of the superfrieze consist in 1's and 0's. More precisely:

$$(2.6) \qquad f_{j,j+m} = 1, \qquad f_{j,j+m+1} = 0, \qquad \varphi_{j,j+m+1} = 0,$$

for all $j \in \mathbb{Z}$ and $m = n - 3$, where $f_{j,i}$ and $\varphi_{j,i}$ are the polynomials defined by Proposition 2.5.4, Part (ii).

Note that equations (2.6), together with the frieze rule, immediately imply $\varphi_{j+\frac{1}{2}, j+m+\frac{3}{2}} = 0$.

It turns out that the algebraic supervarieties of friezes and that of supersymmetric Hill's equations (1.4) can be identified.

Theorem 2.7.2. *The space of superfriezes of width m is an algebraic supervariety isomorphic to the supervariety $\mathcal{E}_n$, for $n = m + 3$.*

Proof. By definition of $f_{j,i}$ and $\phi_{j,i}$, one has for all $i, j \in \mathbb{Z}$,

$$(2.7) \qquad \begin{pmatrix} f_{j,i+n-2} \\ f_{j,i+n-1} \\ \varphi_{j,i+n-1} \end{pmatrix} = M_i \begin{pmatrix} f_{j,i-2} \\ f_{j,i-1} \\ \varphi_{j,i-1} \end{pmatrix},$$

where M_i is as in (1.6).

Given a Hill equation with potential $\beta_i + \xi a_i$, the condition that the monodromy is as in (1.8) implies the relations (2.6), by substituting $i = j - 1$ into (2.7), and using (2.4).

Conversely, assume that the variables $(a_1, \ldots, a_n, \beta_1, \ldots \beta_n)$ satisfy the relations (2.6). Substituting $i = j - 1$ and $i = j$ into (2.7), one obtains

$$M_j \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad M_j \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

for all $j \in \mathbb{Z}$. Hence, M_j is of the form

$$M_j = \left(\begin{array}{cc|c} -1 & 0 & \gamma \\ 0 & -1 & \delta \\ \hline 0 & 0 & e \end{array} \right).$$

Finally, since $M_j \in \mathrm{OSp}(1|2)$, one deduces that M_j is as in (1.8). $\qquad \square$

Proposition 1.4.3 now implies the following.

Corollary 2.7.3. *The space of superfriezes of width m is an algebraic supervariety of dimension $m|m + 1$.*

2.8. **Explicit bijection.** Given a Hill equation (1.4), let us define the corresponding superfrieze. Fix $j \in \mathbb{Z}$, and choose a solution $(V_i^j + \xi W_i^j)$ with the initial conditions $(V_{j-2}^j, W_{j-1}^j, V_{j-1}^j) = (0, 0, 1)$. Form the superfrieze defined by

$$(\varphi_{j,i}, f_{j,i}) := (W_i^j, V_i^j),$$

for all $i, j \in \mathbb{Z}$. Note that the odd entries with half-integer indices are defined by the frieze rule:

$$\varphi_{j-\frac{1}{2}, i+\frac{1}{2}} = f_{j,i} \varphi_{j-1,i} - f_{j-1,i} \varphi_{j,i}.$$

The chosen initial condition implies that $V_{j-3}^j = -1$ and $W_{j-2} = 0$.

The (anti)periodicity condition (1.7) then reads:

$$V_{j+n-3}^j = f_{j,j+n-3} = 1, \qquad W_{j+n-2}^j = \varphi_{j,j+n-2} = 0, \qquad V_{j+n-2}^j = f_{j,j+n-2} = 0.$$

Therefore, we obtain a point of the supervariety of superfriezes.

2.9. **Laurent phenomenon for superfriezes.** The following Laurent phenomenon occurs in the Coxeter friezes: every entries can be expressed as a Laurent polynomial in the entries of any diagonal. Example 2.1.2 illustrates this property. Similar phenomenon occurs in the superfriezes.

Proposition 2.9.1. *Entries of a superfrieze are Laurent polynomials in the entries of any diagonal.*

Proof. Let us fix a South-East diagonal (W_i, V_i) in the superfrieze. By Lemma 2.5.2, one can express the first rows as

$$\beta_i = \frac{W_{i-1} - W_i}{V_{i-1}}, \qquad a_i = \frac{V_i + V_{i-2} - \beta_i W_{i-1}}{V_{i-1}}.$$

Therefore, the first two rows are Laurent polynomials in (W_i, V_i). All the entries of the superfrieze are polynomial in the first two rows. Hence the proposition. $\qquad \square$

3. Open problems

We formulate here a series of problems naturally arising in the study of superfriezes and supersymmetric difference equations.

3.1. **Supervariety $\mathcal{E}_n$.** Our first two problems concern an explicit form of the equations characterizing the supervariety $\mathcal{E}_n$.

Problem 1. *Determine the formula for the entries of a superfrieze.*

In other words, the problem consists in calculating "supercontinuants". The first examples are:

$$K(a_i, \beta_i) = a_i, \qquad K(a_i, a_{i+1}, \beta_i, \beta_{i+1}) = a_i a_{i+1} - 1 + \beta_i \beta_{i+1},$$

$$K(a_i, \dots, \beta_{i+2}) = a_i a_{i+1} a_{i+2} - a_i - a_{i+2} + \beta_i \beta_{i+1} + \beta_i a_{i+1} a_{i+2} + a_i a_{i+1} \beta_{i+2} + \beta_i \beta_{i+2},$$

cf. Example 2.5.5. Is there a determinantal formula (using Berezinians) analogous to the classical continuants?

The next question concerns the odd entries of a superfrieze.

Problem 2. *Do the odd variables β_i of the first odd row satisfy the system of linear equations generalizing the system of Section 1.5?*

Examples considered in Section 1.5 show that for small values of n the variables β_i satisfy linear systems with matrices given by the purely even Coxeter frieze patterns obtained by projection of superfriezes.

In this paper, we do not investigate the geometric meaning of superfriezes and Hill's equations. Recall that classical Hill's equations Coxeter's friezes are related to the moduli spaces $\widehat{\mathcal{M}}_{0,n}$; see Section 1.1 and [26]. We believe that the situation is similar in the supercase.

Problem 3. *Does the algebraic supervariety $\mathcal{E}_n$ contain the supersymmetric moduli space $\mathfrak{M}_{0,n}$ (see [31]) as an open dense subvariety?*

3.2. **Operators of higher orders.** In this paper, we do not consider the general theory of supersymmetric difference operators. We believe that such a theory can be constructed with the help of the shift operator $\mathfrak{T}$; see formula (1.2), and formulate here a problem to develop such a theory in full generality. The corresponding theory of superfriezes must generalize the notion of SL_k-friezes; see [1, 26].

To give an example, we investigate the next interesting case after the Sturm-Liouville operators, namely the operators of order $\frac{5}{2}$. We omit the details of computations.

In the continuous case, the operators we consider are of the form

$$D^5 + F(x, \xi)D + G(x, \xi),$$

where the functions $F(x, \xi)$ and $G(x, \xi)$ are even and odd, respectively.

The discrete version is as follows

$$\mathfrak{T}^5 + \mathfrak{T}^4 + U\mathfrak{T}^3 + V\mathfrak{T}^2 - \Pi,$$

and the corresponding equation written in the matrix form is as follows:

$$\begin{pmatrix} V_{i-2} \\ V_{i-1} \\ V_i \\ W_{i-1} \\ W_i \end{pmatrix} = \left(\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -a_i' & a_i & 0 & \beta_i \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \beta_i' & -1 & a_i' - 1 \end{array} \right) \begin{pmatrix} V_{i-3} \\ V_{i-2} \\ V_{i-1} \\ W_{i-2} \\ W_{i-1} \end{pmatrix},$$

where

$$F_i = a_i' - 1 + \xi(\beta_i + \beta_i'), \qquad G_i = \beta_i' + \xi a_i.$$

The periodicity condition in this case should be:

$$V_{i+n} = V_i, \qquad W_{i+n} = -W_i.$$

We conjecture that the above difference equations correspond to a variant of superfriezes analogous to the 2-friezes; see [25, 24].

<center>APPENDIX: ELEMENTS OF SUPERALGEBRA</center>

To make the paper self-contained, we briefly describe several elementary notions of superalgebra and supergeometry used above. For more details, we refer to the classical sources [3, 18, 20, 21].

Supercommutative algebras. As before, Latin letters denote even variables, and Greek letters the odd ones. Consider algebras of polynomials $\mathbb{K}[x_1, \ldots, x_n, \xi_1, \ldots, \xi_k]$, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$, or some other supercommutative ring, and where x_i are standard commuting variables, while the odd variables ξ_i commute with x_i and anticommute with each other:

$$\xi_i \xi_j = -\xi_j \xi_i,$$

for all i, j; in particular, $\xi_i^2 = 0$. Every supercommutative algebra is a quotient of a polynomial algebra by some ideal. Every supercommutative algebra is the algebra of regular functions on an algebraic supervariety (which can be taken for a definition of the latter notion). Every Lie superalgebra is the algebra of derivations of a supercommutative algebra, for instance, vector fields are derivations of the algebra of regular functions on an algebraic supervariety.

An example of supercommutative algebra is the Grassmann algebra of differential forms on a vector space. Let $(x_1, \ldots, x_n)$ be coordinates, and $(dx_1, \ldots, dx_n)$ their differentials, one replaces all the differentials dx_i by the odd variables ξ_i, to obtain an isomorphic algebra.

We often need to calculate rational functions with odd variables. The main ingredient is the obvious formula $(1 + \xi)^{-1} = 1 - \xi$. For instance, we have:

$$\frac{y}{x + \xi} = \frac{y}{x(1 + \xi/x)} = \frac{y}{x} - \xi \frac{y}{x^2}.$$

The supergroup $\mathrm{OSp}(1|2)$**.** The supergroup $\mathrm{OSp}(1|2)$ is isomorphic to the supergroup of linear symplectic transformations of the $2|1$-dimensional space equipped with the symplectic form

$$\omega = dp \wedge dq + \frac{1}{2} d\tau \wedge d\tau,$$

where p, q, τ are linear coordinates.

The supergroup $\mathrm{OSp}(1|2)$ is the following $3|2$-dimensional supergroup of matrices:

$$\left(\begin{array}{cc|c} a & b & \gamma \\ c & d & \delta \\ \hline \alpha & \beta & e \end{array} \right) \quad \text{such that} \quad \left\{ \begin{array}{rcl} ad - bc & = & 1 - \alpha\beta, \\ e & = & 1 + \alpha\beta, \\ -a\delta + c\gamma & = & \alpha \\ -b\delta + d\gamma & = & \beta. \end{array} \right.$$

For definition, properties and applications of this supergroup; see [20, 14]. Note that the above relations also imply:

$$\gamma = a\beta - b\alpha, \quad \delta = c\beta - d\alpha,$$

and $\alpha\beta = \gamma\delta$.

Left-invariant vector fields on $\mathbb{R}^{1|1}$ and supersymmetric linear differential operators.
Consider the space $\mathbb{R}^{1|1}$ with linear coordinates (x, ξ). We understand the algebra of algebraic functions on this space as the algebra of polynomials in one even and one odd variables:

$$F(x, \xi) = F_0(x) + \xi F_1(x),$$

where F_0 and F_1 are usual polynomials in x.

The following two vector fields

$$X = \frac{\partial}{\partial x}, \qquad D = \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial x}$$

are important in superalgebra and supergeometry. These vector fields are left-invariant with respect to the supergroup structure on $\mathbb{R}^{1|1}$ given by

$$(r, \lambda) \cdot (s, \mu) = (r + s + \lambda\mu, \, \lambda + \mu).$$

Moreover, X and D are characterized (up to a multiple) by the property of left-invariance, as the only even and odd left-invariant vector fields on $\mathbb{R}^{1|1}$, respectively.

The vector fields X and D form a $1|1$-dimensional Lie superalgebra since

$$D^2 = \frac{1}{2}[D, D] = -X,$$

and $[X, D] = 0$, with one odd generator D.

The space $\mathbb{R}^{1|1}$ equipped with the vector field D is often called by physicists the one-dimensional "superspacetime". A *supersymmetric* differential operator on $\mathbb{R}^{1|1}$ is an operator that can be expressed as a polynomial in D.

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