

A STEP TOWARDS CLUSTER SUPERALGEBRAS

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ABSTRACT. We introduce a class of commutative superalgebras generalizing cluster algebras. A cluster superalgebra is defined by a hypergraph called an “extended quiver”, and transformations called mutations. We prove the super analog of the “Laurent phenomenon”, i.e., that all elements of a given cluster superalgebra are Laurent polynomials in the initial variables, and find an invariant presymplectic form. Examples of cluster superalgebras are provided by superanalogs of Coxeter’s frieze patterns. We apply the Laurent phenomenon to construct a new integer sequence extending the Somos-4 sequence.

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Key words and phrases. Cluster algebra, Supercommutative algebra, Quiver mutations, Integer sequences, Dual numbers.

INTRODUCTION

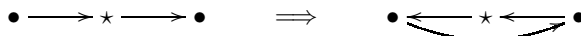
Commutative superalgebras appear naturally in classical geometry. One classical example is the classical Grassmann algebra of differential forms on the manifold. Commutative superalgebras are an essential part of any cohomology theory. Lie superalgebras also appear in physics where they are extensively used to describe symmetries of elementary particles. Theory of supermanifolds was developed in 70's by Berezin and Leites, see [5, 10, 30] as well as [27, 29, 28]. This theory offers geometry and analysis based on spaces ringed by superalgebras. Commutative superalgebras can be regarded as a special class of non-commutative or “quantum” algebras, their main feature is the presence of odd, or “Grassmann” elements that square to zero.

Classical cluster algebras. Cluster algebras, discovered by Fomin and Zelevinsky [14], are a special class of commutative associative algebras. A cluster algebra is a subalgebra of the algebra generated by Laurent polynomials with positive integer coefficients in $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. A cluster algebra is usually defined with the help of a quiver (an oriented graph) with no loops and no 2-cycles; the generators of the algebra are defined with the help of *exchange relations* and *mutations* of the initial quiver. More precisely, the vertices of the initial quiver are labeled by $\{x_1, \dots, x_n\}$; the mutation at a chosen variable x_k changes the quiver and replaces x_k with the new variable

$$x'_k = \frac{M_1 + M_2}{x_k},$$

where M_1 and M_2 are the monomials obtained as products of variables connected to x_k by ingoing and outgoing arrows, respectively. The above formula is called an exchange relation; a cluster algebra is generated by all possible mutations and exchange relations.

Mutations of the quiver at a chosen vertex are transformations that can be illustrated by the following diagram:



The most important point is the creation of a new arrow¹.

Cluster algebras naturally appear in algebra, geometry and combinatorics. The algebras of regular functions of many algebraic varieties, related to Lie theory, have cluster structure. The main examples are simple Lie groups [2, 20], Grassmannians [39], flag varieties, etc. Cluster algebras are closely related to symplectic and Poisson geometry [19, 21], representation theory, Teichmüller spaces [12] and various moduli spaces. For more information and references, see surveys [31, 40].

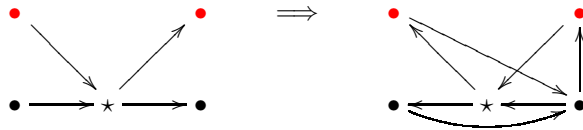
Cluster algebras are also closely related to integrable systems. The Hirota (or octahedral) recurrence, see [42], which is a famous and “universal” discrete integrable system, is an example of cluster exchange relations. Many spaces on which discrete integrable systems act, such as moduli spaces of configurations of points in projective spaces, have cluster structures, see [36, 35]. Among a wealth of notions related to these moduli spaces, we mention frieze patterns, see [9, 7]. Moduli spaces of configurations of points are also related to the spaces of linear difference equations with special periodicity conditions; for a detailed account, see [35]. For recent developments in this area, see [13, 18, 22, 17, 33, 23, 38, 25].

Towards cluster superalgebras. A challenging problem is to develop a notion of cluster superalgebra. This paper is a first step in this direction. The general idea is to include odd, or Grassmann, indeterminates $\{\xi_1, \dots, \xi_m\}$, that anticommute with each other, and in particular, square to zero. The cluster superalgebra is a certain subalgebra of

$$\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, \xi_1, \dots, \xi_m].$$

¹ Exactly the same mutation rules are intensively used in physics under the name of Seiberg Duality, cf. [4], for a recent treatment, see [1] and references therein.

A cluster superalgebra is defined by an “extended quiver” which is a hypergraph extending a classical quiver. The main ingredients are modified exchange relations and quiver mutations. The vertices of the classical quiver are labeled by the even variables, the new vertices are labeled by the odd variables. Essentially, the mutations of an extended quiver (observe that the two upper vertices are colored while the three lower ones are black) are defined by the following rule:



Note that this rule has similarity with mutation of colored graphs, see [6, 32].

Comments. It should be stressed that the algebras constructed in this paper are not particular cases of known versions of cluster algebras. In particular a cluster super algebra is not a quantum cluster algebra [3], or other known non-commutative analogs of cluster algebras.

We believe that the definition of cluster superalgebra can be more general. The main limitations of our construction are:

- absence of arrows between odd vertices;
- lack of exchange relations for the odd variables;
- lack of geometric examples.

It would be interesting to look for more general definitions, as well as for more examples of cluster superalgebras and related supermanifolds. In particular, it would be interesting to look for structures of cluster superalgebra on the space of regular functions of classical supergroups, similar to cluster algebra structures of [2].

It would also be interesting to look for examples related to super Lie theory. However, a naive attempt to use Berezinians and condensation formulas encounters some difficulties. The series of supergroups $SL(n|m)$ lead to the classical cluster algebras, while the case of $OSp(n|m)$ is much more difficult.

Our approach is based on relations of cluster algebras to moduli spaces, linear difference equations, and frieze patterns. Supersymmetric linear difference equations with some special monodromy conditions and the corresponding *superfriezes*, analogous to Coxeter’s frieze patterns, were recently introduced in [37]. Our definition of cluster superalgebras suggested in the present paper is a result of analysis of superfriezes. We consider a class of examples of cluster superalgebras “of type A ” that correspond to superfriezes.

Structure of the paper. The paper consists of seven sections.

In introductory Section 1, we briefly review classical cluster algebras, and show how to recover the rules of mutation of a quiver from the canonical presymplectic form and exchange relations.

In Section 2, we introduce the notion of extended quiver and the rules of mutation.

In Section 3, we define the exchange relations generalizing the classical exchange relations to the presence of odd variables. We also introduce the presymplectic form and formulate the main results of the paper: the Laurent phenomenon and invariance of the presymplectic form. We finally define cluster superalgebras.

In Section 4, we consider a list of examples.

In Section 5, we present an infinite series of examples of cluster superalgebras, related to the notion of “superfriezes” and difference equations studied in [37]. Observe once more that superfriezes is the origin of the general notion of cluster superalgebra developed in the present paper.

In Section 6, we prove the results formulated in Section 3.

In Sections 7 and 8, we present number theoretic applications.

1. CLASSICAL CLUSTER ALGEBRAS AND THEIR PRESYMPLECTIC FORM

This section is a very elementary introduction to cluster algebras; our main goal is to explain the role of the canonical presymplectic 2-form.

A *cluster algebra of rank n* is a commutative associative algebra associated to a *quiver*, i.e., a finite oriented graph, \mathcal{Q} , with no loops and no 2-cycles. We use the common notation: \mathcal{Q}_0 is the set of vertices, and \mathcal{Q}_1 is the set of arrows of \mathcal{Q} . The vertices of \mathcal{Q} are labeled by commuting variables $\{x_1, \dots, x_n\}$, where $|\mathcal{Q}_0| = n$.

1.1. The definition. The cluster algebra defined by \mathcal{Q} is generated by $\{x_1, \dots, x_n\}$ and the additional rational functions x'_k, \dots obtained recursively by a sequences of transformations called *mutations*:

$$\mu_k : (\{x_1, \dots, x_n\}, \mathcal{Q}) \mapsto (\{x_1, \dots, x_n\} \setminus \{x_k\} \cup \{x'_k\}, \mathcal{Q}').$$

For every $k \in \{1, \dots, n\}$, the mutation at vertex x_k replaces the variable x_k by the new function x'_k defined by the formula:

$$x_k x'_k = \prod_{x_k \rightarrow x_j} x_j + \prod_{x_i \rightarrow x_k} x_i,$$

where the products are taken over the set of arrows $(x_i \rightarrow x_k) \in \mathcal{Q}_1$ and $(x_k \rightarrow x_j) \in \mathcal{Q}_1$, respectively (with fixed k). The above formula is called an *exchange relation*.

The mutation of the quiver $\mu_k : \mathcal{Q} \rightarrow \mathcal{Q}'$ at the vertex x_k is defined according to the following three rules:

- (1) for every path $(x_i \rightarrow x_k \rightarrow x_j)$ in \mathcal{Q} , add an arrow $(x_i \rightarrow x_j)$;
- (2) reverse all the arrows leaving or arriving at x_k ;
- (3) remove all 2-cycles created by rule (1).

Every mutation is an involution, i.e., $\mu_k^2 = \text{Id}$, but two different mutations do not necessarily commute. Applying all possible mutations, one obtains (possibly infinitely many) different quivers, and rational functions x'_k, \dots in the initial coordinates $\{x_1, \dots, x_n\}$. The cluster algebra associated with the initial quiver \mathcal{Q} , is the associative commutative algebra generated by all the functions obtained in the mutation process.

Some of the initial cluster coordinates are called *frozen*, and remain unchanged. One simply does not consider mutations at frozen coordinates. These coordinates are additional parameters of the cluster algebra.

Among the main results of Fomin and Zelevinsky are:

- *Laurent phenomenon*: all elements of a given cluster algebra are Laurent polynomials in $\{x_1, \dots, x_n\}$, see [14],[15].
- *A finite type classification*: a cluster algebra is of finite type, i.e., finitely generated, if and only if the quiver \mathcal{Q} is equivalent under a sequence of mutations to one of the Dynkin quivers of type A, D , or E , see [16].

Furthermore, the *positivity property* conjectured in [14] and recently proved in [26], states that the coefficients of the Laurent monomials in all the generators of a cluster algebra are non-negative integers.

1.2. The canonical presymplectic form. Every cluster algebra (and thus every algebraic manifold whose algebra of regular functions has a cluster algebra structure) has a canonical presymplectic differential 2-form introduced by Gekhtman, Shapiro and Vainshtein [19]. It is given by

$$\omega = \sum_{x_i \rightarrow x_j} \frac{dx_i \wedge dx_j}{x_i x_j},$$

where the summation is performed over the the arrows $(x_i \rightarrow x_k) \in \mathcal{Q}_1$. This form is invariant with respect to the cluster mutations and therefore is well-defined globally, on every cluster variety. Note that the form ω is obviously closed since it is with constant coefficients in coordinates $\log x_i$. However, it is not necessarily non-degenerate, for instance, if n is odd.

Quite remarkably, among the following three data:

$$(\text{exchange relations,} \quad \text{quiver mutations,} \quad \text{canonical 2-form } \omega)$$

every two contain the full information, and allow one to recover the third ingredient.

Let us show how to recover the quiver mutations from the exchange relations and the invariant 2-form ω . In order to make the exposition elementary, we first consider a simple example.

Example 1.1. Consider the cluster algebra associated to the Dynkin quiver A_3 :

$$x \longrightarrow y \longrightarrow z$$

The corresponding 2-form is

$$\omega = \frac{dx \wedge dy}{xy} + \frac{dy \wedge dz}{yz},$$

and the exchanging relation for y is $y' = (x+z)/y$. Expressing then $y = (x+z)/y'$, and substituting into ω , one obtains from the Leibniz rule:

$$\begin{aligned} \omega &= \frac{dx \wedge dz}{x(x+z)} - \frac{dx \wedge dy'}{xy'} + \frac{dx \wedge dz}{(x+z)z} - \frac{dy' \wedge dz}{y'z} \\ &= \frac{dx \wedge dz}{xz} + \frac{dy' \wedge dx}{xy'} + \frac{dz \wedge dy'}{y'z}. \end{aligned}$$

Invariance of ω means that, after the above coordinate transformation, ω is exactly the form of the quiver

$$x \xleftarrow{\quad} y' \xleftarrow{\quad} z$$

obtained as a result of the mutation at the vertex y .

The general situation is similar. Rewrite the form ω using the summation over vertices of \mathcal{Q} . One immediately obtains from the Leibniz rule:

$$\omega = \frac{1}{2} \sum_{x_k \in \mathcal{Q}_0} \frac{d \frac{x_{I_k}}{x_{J_k}} \wedge dx_k}{\frac{x_{I_k}}{x_{J_k}} x_k},$$

where

$$x_{I_k} = x_{i_1} \cdots x_{i_s}, \quad x_{J_k} = x_{j_1} \cdots x_{j_t}$$

are the monomials obtained as the products of coordinates connected to x_k by ingoing arrows: $(x_i \rightarrow x_k)$, and by outgoing arrows: $(x_k \rightarrow x_j)$, respectively.

Invariance of the latter formula for ω under mutations at x_k is quite obvious. Indeed, temporarily using the notation $\widetilde{x}_k' := \frac{x_k'}{x_{J_k}}$, rewrite the exchange relations in the form:

$$x_k = \frac{1}{\widetilde{x}_k'} \left(1 + \frac{x_{I_k}}{x_{J_k}} \right).$$

Let also

$$\omega_{x_k} = \frac{1}{2} \frac{d \frac{x_{I_k}}{x_{J_k}} \wedge dx_k}{\frac{x_{I_k}}{x_{J_k}} x_k}$$

be the term of ω containing x_k . Substituting the above equation for x_k , and by the Leibniz rule one immediately obtains:

$$\begin{aligned}\omega_{x_k} &= \frac{1}{2} \frac{d\widetilde{x}_k' \wedge d\frac{x_{I_k}}{x_{J_k}}}{\widetilde{x}_k' \frac{x_{I_k}}{x_{J_k}}} \\ &= \frac{1}{2} \left(\frac{dx_k' \wedge d\frac{x_{I_k}}{x_{J_k}}}{x_k' \frac{x_{I_k}}{x_{J_k}}} - \frac{dx_{J_k} \wedge dx_{I_k}}{x_{I_k} x_{J_k}} \right).\end{aligned}$$

The first term corresponds to the reversed arrows at x_k' , while the second term corresponds to the extra arrows ($x_j \rightarrow x_i$). Invariance of ω therefore implies the rules (1) and (2) of quiver mutations.

The Gekhtman-Shapiro-Vainshtein form ω , and related family of Poisson structures, play an important role in applications. It equips cluster algebraic varieties with additional geometric structures. A similar presymplectic form will be crucial for us to understand mutations of extended quivers.

2. EXTENDED QUIVERS AND THEIR MUTATIONS

We introduce extended quivers, and describe their mutation rules. It turns out that an extended quiver is not a graph but an oriented *hypergraph*. More precisely, given a quiver \mathcal{Q} , we add new, colored (or odd), vertices, and complete the set of edges of \mathcal{Q} by adding some 2-paths joining three vertices.

The reason for this notion is the general idea of superalgebra and supergeometry, that supersymmetric version of every object should be understood as its “square root”. The notion of extended quiver is an attempt to apply this idea in combinatorics: a square root of an edge in a graph is understood as a 2-path joining two odd vertices through an even vertex.

2.1. Introducing extended quivers.

Definition 2.1. *Given a quiver \mathcal{Q} with no loops and no 2-cycles, an extended quiver $\widetilde{\mathcal{Q}}$ with underlying quiver \mathcal{Q} , is an oriented hypergraph defined as follows.*

- (A) *The vertices of \mathcal{Q} are labeled by $\{x_1, \dots, x_n\}$, $\widetilde{\mathcal{Q}}$ has m extra “colored” vertices labeled by the odd variables $\{\xi_1, \dots, \xi_m\}$, so that*

$$\widetilde{\mathcal{Q}}_0 = \mathcal{Q}_0 \cup \{\xi_1, \dots, \xi_m\}.$$

- (B) *Some of the new vertices $\{\xi_1, \dots, \xi_m\}$ are related by 2-paths through the vertices $\{x_1, \dots, x_n\}$ of the underlying quiver \mathcal{Q} . More precisely, for every $1 \leq k \leq n$, two non-intersecting subsets*

$$I_k, J_k \subset \{1, \dots, m\}, \quad I_k \cap J_k = \emptyset$$

are fixed; the set of arrows \mathcal{Q}_1 is completed by the set of 2-paths:

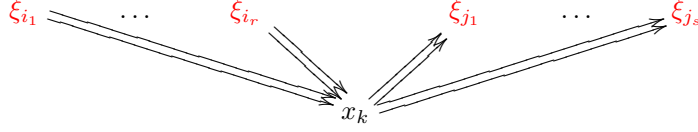
$$\widetilde{\mathcal{Q}}_1 = \mathcal{Q}_1 \cup_k \{(\xi_i \rightarrow x_k \rightarrow \xi_j), i \in I_k, j \in J_k\},$$

so that the following condition is satisfied.

- (C) *Every vertex ξ_i , for $i \in I_k$, is connected to every vertex ξ_j , for $j \in J_k$, by exactly the same number of 2-paths through x_k .*

Although $\widetilde{\mathcal{Q}}$ is a hypergraph, and therefore can hardly be represented graphically, the above definition is illustrated by the following diagram. To fix the notation, let $I_k = \{i_1, \dots, i_r\}$, and

$$J_k = \{j_1, \dots, j_s\}.$$



Here, every vertex $\xi_i \in \{\xi_{i_1}, \dots, \xi_{i_r}\}$ is connected to every vertex $\xi_j \in \{\xi_{j_1}, \dots, \xi_{j_s}\}$ by a certain number of 2-paths $(\xi_i \rightarrow x_k \rightarrow \xi_j)$ through x_k . This number is independent of the choice of $i \in I_k$ and $j \in J_k$.

Remark 2.2. (a) Condition (C) is not just a technical restriction. Our main results: the Laurent phenomenon and invariant presymplectic form, hold if and only if this condition is satisfied.

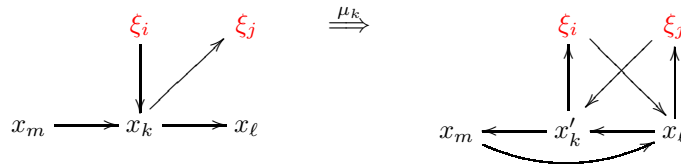
(b) Note that we do not consider arrows between the odd vertices of $\tilde{\mathcal{Q}}$, and this is certainly an interesting question whether one can add such arrows and create a more rich combinatorics of extended quivers.

2.2. Quiver mutations. Let us define the mutation rules of an extended quiver. These mutations are performed at even vertices. The rules of mutations are modifications of the classical rules (1), (2) (3), see Section 1.1.

Definition 2.3. Given an extended quiver $\tilde{\mathcal{Q}}$ and an even vertex $x_k \in \mathcal{Q}_0$, the mutation μ_k is defined by the following rules:

- (0) The underlying quiver $\mathcal{Q} \subset \tilde{\mathcal{Q}}$ mutates according to the same rules (1), (2), (3) as in the classical case. The k th label changes to x'_k (whose new value will be defined later on).
- (1*) Given a 2-path $(\xi_i \rightarrow x_k \rightarrow \xi_j) \in \tilde{\mathcal{Q}}_1$, for all $x_\ell \in \mathcal{Q}_0$ connected to x_k by an outgoing arrow $(x_k \rightarrow x_\ell)$, add the 2-paths $(\xi_i \rightarrow x_\ell \rightarrow \xi_j)$.
- (2*) Reverse all the 2-paths through x_k , i.e., change $(\xi_i \rightarrow x_k \rightarrow \xi_j)$ to $(\xi_i \leftarrow x_k \leftarrow \xi_j)$.
- (3*) Remove two-by-two the 2-paths through x_k (if any) with opposite orientations, created by rule (1*), i.e., 2-paths $(\xi_i \rightarrow x_\ell \rightarrow \xi_j)$ and $(\xi_i \leftarrow x_\ell \leftarrow \xi_j)$ cancel each other.

The above rules can be illustrated by the diagram:



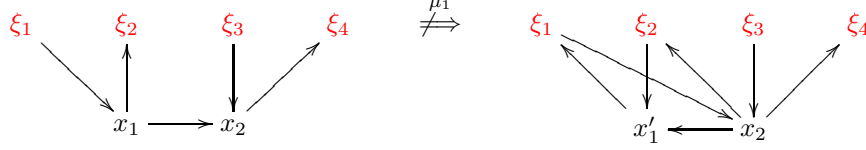
Remark 2.4. To compare the new rule (1*) to the classical mutation rule (1), note that the new rule changes the set $\tilde{\mathcal{Q}}_1$ of 2-paths, and not just the set of arrows, as the old rule (1).

2.3. Allowed and forbidden mutations. A mutation of the extended quiver $\tilde{\mathcal{Q}}$ at a vertex $x_k \in \mathcal{Q}_0$ is *allowed* if, in the resulting quiver $\mu_k(\tilde{\mathcal{Q}})$ satisfies Condition (C) from Section 2.1. The reason why some mutations can violate this condition, and are therefore forbidden, is due to the fact that the sets I_ℓ and J_ℓ attached to a vertex x_ℓ connected to x_k by an arrow $(x_k \rightarrow x_\ell)$ are transformed:

$$\mu_k : \begin{cases} I_\ell \Rightarrow I_\ell \cup I_k, \\ J_\ell \Rightarrow J_\ell \cup J_k, \end{cases}$$

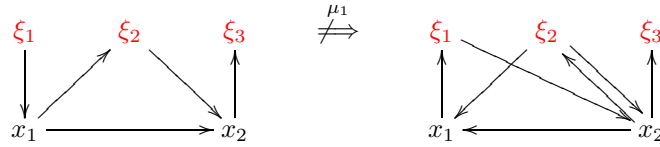
as a result of the rule (1*).

Example 2.5. a) Mutation of the following quiver at x_1 is not allowed.



Indeed, this mutation would produce an extended quiver \tilde{Q} such that \tilde{Q}_1 contains exactly two 2-paths through x_2 , namely $(\xi_1 \rightarrow x_2 \rightarrow \xi_2)$ and $(\xi_3 \rightarrow x_2 \rightarrow \xi_4)$. Therefore in this case we have: $I_2 = \{1, 3\}, J_2 = \{2, 4\}$. However, ξ_1 is not connected to ξ_4 and ξ_3 is not connected to ξ_2 by a 2-path through x_2 , which violates Condition (C).

b) The mutation of the following quiver at x_1 is not allowed



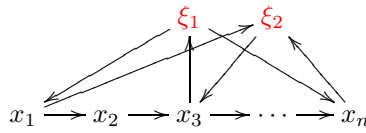
since the subsets I_2 and J_2 are not well-defined.

c) The following mutation is allowed



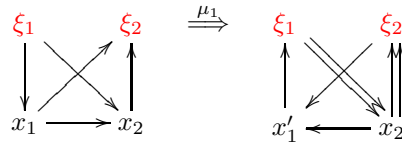
Indeed, in the resulting extended quiver, $I_2 = \{1, 3\}, J_2 = \{2\}$, and \tilde{Q}_1 contains two 2-paths through x_2 , namely $(\xi_1 \rightarrow x_2 \rightarrow \xi_2)$ and $(\xi_3 \rightarrow x_2 \rightarrow \xi_2)$, so that Condition (C) is satisfied.

2.4. Extended quivers with two colored vertices. Consider an important class of extended quivers that have two colored vertices. This is the first non-trivial example that already leads to very interesting cluster superalgebras, see Section 4.



In the case of two colored vertices, *all mutations are allowed*.

Example 2.6. For instance, one has



This mutation creates a new 2-path $(\xi_1 \rightarrow x_2 \rightarrow \xi_2)$, so that the resulting extended quiver has two such paths.

3. EXCHANGE RELATIONS AND PRESYMPLECTIC FORM

We define the exchange relations of the even variables $\{x_1, \dots, x_n\}$ corresponding to mutations of extended quivers defined in the previous section. We introduce a presymplectic form invariant with respect to mutations of the extended quiver and the corresponding exchange relations. We also formulate two main results of the paper: the Laurent phenomenon for the exchange relations, and the invariance property of the presymplectic form.

We finally define our notion of cluster superalgebra, as the algebra generated by the variables $\{x_1, \dots, x_n, \xi_1, \dots, \xi_m\}$ called the “initial seed”, and all their mutations x'_k, x''_k, \dots defined by the exchange relations. Note that every cluster superalgebra has an *underlying* classical cluster algebra, obtained by substituting $\xi_i \equiv 0$.

3.1. Exchange relations.

Definition 3.1. *Given an extended quiver $\tilde{\mathcal{Q}}$, the mutation μ_k replaces the variable x_k by a new variable, x'_k , other variables remain unchanged:*

$$\mu_k : \{x_1, \dots, x_n, \xi_1, \dots, \xi_m\} \rightarrow \{x_1, \dots, x_n, \xi_1, \dots, \xi_m\} \setminus \{x_k\} \cup \{x'_k\}.$$

The new variable is defined by the following formula

$$(1) \quad x_k x'_k = \prod_{x_k \rightarrow x_j} x_j + \left(1 + \sum_{\xi_i \rightarrow x_k \rightarrow \xi_j} \xi_i \xi_j\right) \prod_{x_i \rightarrow x_k} x_i,$$

that will be called, as in the classical case, an exchange relation.

Note that, after substitution $\xi \equiv 0$, the above formula obviously coincides with the exchange relations for the classical cluster algebra corresponding to the underlying quiver \mathcal{Q} . The first summand in (1) is exactly as in the classical case, the second one is modified.

Our first main result is the Laurent phenomenon. Let us emphasise that, since the division by odd coordinates is not well-defined, all the Laurent polynomials we consider have denominators equal to some monomials in $\{x_1, \dots, x_n\}$.

Theorem 1. *For every extended quiver $\tilde{\mathcal{Q}}$, all the rational functions x'_k, x''_k, \dots , obtained recurrently by any series of consecutive admissible mutations, are Laurent polynomials in the initial coordinates $\{x_1, \dots, x_n, \xi_1, \dots, \xi_m\}$.*

This theorem is proved in Section 6.1.

Remark 3.2. Note that, unlike the purely even case, the above mutation of x_k is *not an involution*, i.e., if the function x''_k is obtained by the iterated mutation μ_k^2 at x_k , then $x''_k \neq x_k$. Using (1) twice, one easily obtains:

$$x''_k = x_k \left(1 - \sum_{\xi_i \rightarrow x_k \rightarrow \xi_j} \xi_i \xi_j\right).$$

It is important to notice however, that this expression belongs to the algebra generated by the initial variables $\{x_k, \xi_1, \dots, \xi_m\}$. Note also that, conversely, $x_k = x''_k \left(1 + \sum_{\xi_i \rightarrow x_k \rightarrow \xi_j} \xi_i \xi_j\right)$.

3.2. The presymplectic form. Consider the following differential 2-form:

$$(2) \quad \omega = \sum_{x_i \rightarrow x_j} \frac{dx_i \wedge dx_j}{x_i x_j} + \sum_{\xi_i \rightarrow x_\ell \rightarrow \xi_j} \frac{d(\xi_i \xi_j) \wedge dx_\ell}{x_\ell}.$$

Note that the summation is organized over the elements of $\tilde{\mathcal{Q}}_1$. The first summand is nothing but the classical presymplectic form.

It turns out that the 2-form (2) is invariant under mutations μ_k . The following statement is our second main result.

Theorem 2. *For every extended quiver $\tilde{\mathcal{Q}}$, the form ω is invariant under mutations μ_k , combined with the exchange relations (1).*

This theorem is proved in Section 6.2

In other words, expressing x_k in terms of the other variables and x'_k , and substituting to (2), one obtains precisely the presymplectic form associated to the extended quiver $\mu_k(\tilde{\mathcal{Q}})$. Geometrically speaking, the above statement means that the form (2) is well-defined on an algebraic supermanifold whose space of regular functions has a structure of cluster superalgebra.

Moreover, as in the purely even case, formulas (1) and (2) allow us to recover the rules of quiver mutations from Definition 2.3.

Remark 3.3. The 2-form ω given by (2) is not the unique invariant presymplectic form. Indeed, adding the (symmetric in ξ) terms of the form $d\xi_i \wedge d\xi_j$, for some i, j , one obtains a large family of presymplectic forms. This is probably equivalent to developing of the theory of mutations at colored vertices and exchange relations of odd variables. Only adding such terms one can expect the 2-form to be non-degenerate when the underlying form is non-degenerate.

3.3. Definition of the cluster superalgebra. Given an extended quiver $\tilde{\mathcal{Q}}$, we call the *cluster superalgebra* associated to $\tilde{\mathcal{Q}}$, the supercommutative superalgebra generated by the initial variables $\{x_1, \dots, x_n, \xi_1, \dots, \xi_m\}$, together with the functions x'_k, x''_k, \dots that can be obtained recursively by sequences of admissible mutations. This superalgebra will be denoted by $A(\tilde{\mathcal{Q}})$.

The pair $(\tilde{\mathcal{Q}}, \{x_1, \dots, x_n, \xi_1, \dots, \xi_m\})$ is called the initial *seed* of the cluster superalgebra.

Given an extended quiver $\tilde{\mathcal{Q}}$, following [14], we call the *exchange graph* the graph \tilde{T} whose vertices are the initial seed $(\tilde{\mathcal{Q}}, \{x, \xi\})$ and all its allowed mutations; the edges are labeled by the numbers from 1 to n . The graph \tilde{T} has no natural orientation. The projection $\tilde{\mathcal{Q}} \mapsto \mathcal{Q}$ defines the projection $\tilde{T} \mapsto T$, where T is the usual exchange graph.

4. ELEMENTARY EXAMPLES

In this section, we consider a number of concrete examples, in order to illustrate the exchange relations (1) and the mutation rules of extended quivers. Most of our examples correspond to extended quivers with two colored vertices. We observe a failure of 5-periodicity that occurs in the classical case.

We show that the Laurent phenomenon holds for all examples we consider, and sometimes simplifications are quite spectacular. We also check that the corresponding presymplectic form is, indeed, invariant in each of our examples.

A brief tutorial on calculus with odd variables. One needs to know a few simple rules in order to work efficiently with odd variables.

Odd variables commute with the even variables and anticommute with each other. One has: $\xi_1 \xi_2 = -\xi_2 \xi_1$. In particular, odd variables square to zero: $\xi_i^2 = 0$.

We will often need to calculate rational functions with odd variables in the denominator. Clearly, division by the odd variables is not well-defined. However, the obvious formula $(1 + \xi)^{-1} = 1 - \xi$ makes many calculations possible. For instance,

$$\frac{y}{x + \xi_1 \xi_2} = \frac{y}{x(1 + \xi_1 \xi_2/x)} = \frac{y}{x} - \xi_1 \xi_2 \frac{y}{x^2}.$$

Example 4.1. Our most elementary example is the extended quiver with one ordinary and two colored vertices. Let us first perform consecutive mutations at the even vertex:



The variable x obeys the following exchange relation:

$$x' = \frac{2}{x} + \frac{\xi_1 \xi_2}{x}, \quad x'' = \frac{2}{x'} - \frac{\xi_1 \xi_2}{x'} = x \left(\frac{2}{2 + \xi_1 \xi_2} - \frac{\xi_1 \xi_2}{2 + \xi_1 \xi_2} \right) = x(1 - \xi_1 \xi_2).$$

Continuing the process, we obtain:

$$x''' = \frac{2}{x} + 3 \frac{\xi_1 \xi_2}{x}, \quad x^{IV} = x(1 - 2\xi_1 \xi_2), \quad \dots$$

As already mentioned, these mutations are not involutions, and the resulting process is infinite and aperiodic.

The corresponding cluster superalgebra is $\mathbb{C}[x^{\pm 1}, \xi_1, \xi_2] \cong \mathbb{C}[x^{\pm 1}] \otimes \mathbb{C}[\xi_1, \xi_2]$.

The presymplectic form corresponding to this extended quiver is:

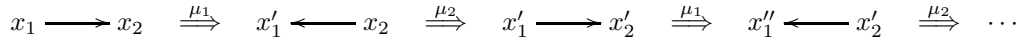
$$\omega = \frac{d(\xi_1 \xi_2) \wedge dx_1}{x_1}.$$

Substituting $x = \frac{2}{x'} + \frac{\xi_1 \xi_2}{x'}$, and using the fact that $d(\xi_1 \xi_2) \wedge d(\xi_1 \xi_2) = 0$, one has:

$$\omega = -\frac{d(\xi_1 \xi_2) \wedge dx'_1}{x'_1},$$

which corresponds to the reversing of the orientation of the 2-path $(\xi_1 \rightarrow x \rightarrow \xi_2)$ by the mutation at x .

Example 4.2. Recall (see [14]) that, in the purely even case of the Dynkin quiver $\mathcal{Q} = A_2$, the process of consecutive mutations:



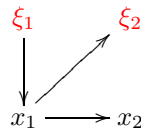
is 5-periodic:

$$x'_1 = \frac{1 + x_2}{x_1}, \quad x'_2 = \frac{1 + x_1 + x_2}{x_1 x_2}, \quad x''_1 = \frac{1 + x_1}{x_2}, \quad x''_2 = x_1, \quad x'''_1 = x_2.$$

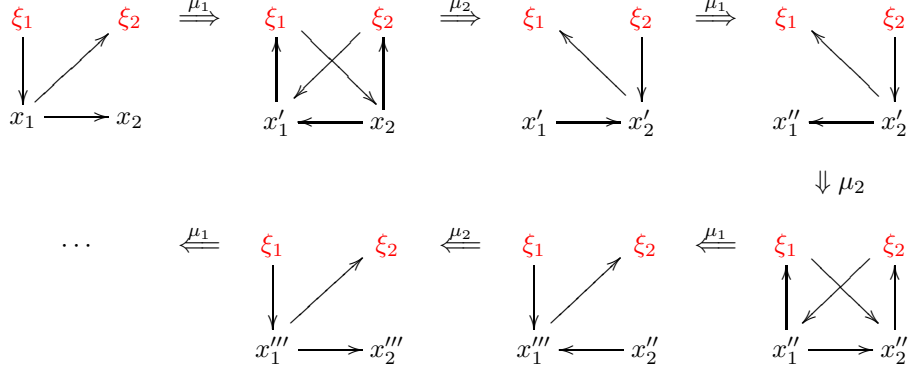
The corresponding exchange graph is a pentagon, and the cluster algebra of A_2 is isomorphic to $\mathbb{C}[x_1, x_2, x'_1, x'_2, x''_1]$.

In the next three examples, we consider different quivers that are extensions of the quiver A_2 .

Example 4.3. Consider the following quiver with two even and two odd coordinates $\{x_1, x_2, \xi_1, \xi_2\}$:



Let us alternate consecutive mutations at even vertices:



Let us give some details of the computations. The first two mutations are obtained immediately from (1):

$$x'_1 = \frac{1+x_2}{x_1} + \frac{\xi_1 \xi_2}{x_1}, \quad x'_2 = \frac{1+x_1+x_2}{x_1 x_2} + \frac{1+x_1}{x_1 x_2} \xi_1 \xi_2.$$

Again, applying the exchange relations (1), and performing simplifications, one obtains:

$$\begin{aligned} x''_1 &= \frac{x'_2+1}{x'_1} \\ &= \frac{1}{x_2} \left(\frac{1+x_1+x_2+x_1 x_2}{1+x_2+\xi_1 \xi_2} + \frac{(1+x_1)\xi_1 \xi_2}{1+x_2+\xi_1 \xi_2} \right) \\ &= \frac{1}{x_2} \left(\frac{1+x_1+x_2+x_1 x_2}{1+x_2} - \frac{1+x_1+x_2+x_1 x_2}{1+x_2} \frac{\xi_1 \xi_2}{1+x_2} + \frac{1+x_1}{1+x_2} \xi_1 \xi_2 \right) \\ &= \frac{1+x_1}{x_2}. \end{aligned}$$

Note that simplifications of the denominators are quite surprising.

Similarly, after some computations, one obtains:

$$x''_2 = x_1(1 - \xi_1 \xi_2), \quad x'''_1 = x_2(1 - \xi_1 \xi_2), \quad x'''_2 = \frac{1+x_2}{x_1} + \frac{\xi_1 \xi_2}{x_1},$$

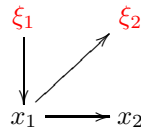
and the next values are:

$$x_1^{IV} = \frac{1+x_1+x_2}{x_1 x_2} (1 + \xi_1 \xi_2) + \frac{1+x_1}{x_1 x_2} \xi_1 \xi_2, \quad x_2^{IV} = \frac{1+x_1}{x_2} (1 + \xi_1 \xi_2), \quad \dots$$

Remarkably enough, all of the coordinates we obtain by iterating the process, are Laurent polynomials. However, the above process is infinite and aperiodic.

Our mutation process is aperiodic and very different from the classical mutation process of the quiver A_2 . However, analyzing the above formulas, we see the following. The corresponding cluster superalgebra is generated by five even and two odd variables: $\{x_1, x_2, x'_1, x'_2, x''_1, \xi_1, \xi_2\}$. Indeed, all the other mutations of x_1 and x_2 are algebraic expressions in these functions.

The presymplectic form corresponding to the initial extended quiver



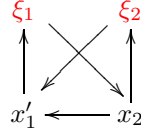
is as follows:

$$\omega = \frac{dx_1 \wedge dx_2}{x_1 x_2} + \frac{d(\xi_1 \xi_2) \wedge dx_1}{x_1}.$$

Let us substitute to this form $x_1 = \frac{1}{x'_1} (1 + x_2 + \xi_1 \xi_2)$. One readily gets:

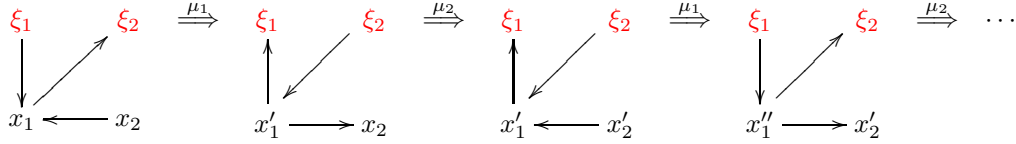
$$\begin{aligned} \omega &= -\frac{dx'_1 \wedge dx_2}{x'_1 x_2} - \frac{d(\xi_1 \xi_2) \wedge dx'_1}{x'_1} + \frac{d(\xi_1 \xi_2) \wedge dx_2}{1 + x_2 + \xi_1 \xi_2} + \frac{d(\xi_1 \xi_2) \wedge dx_2}{(1 + x_2 + \xi_1 \xi_2)x_2} \\ &= -\frac{dx'_1 \wedge dx_2}{x'_1 x_2} - \frac{d(\xi_1 \xi_2) \wedge dx'_1}{x'_1} + \frac{d(\xi_1 \xi_2) \wedge dx_2}{x_2}. \end{aligned}$$

The latter expression is precisely the presymplectic form corresponding to



which is the result of the mutation μ_1 . Hence, ω is invariant under this mutation. A similar computation shows that ω is also invariant under the mutation μ_2 , etc.

Example 4.4. Starting with a similar quiver and performing the same sequence of mutations, one has:



and then again μ_1, μ_2 , etc. One then obtains, after computations:

$$x'_1 = \frac{1 + x_2}{x_1} + \frac{x_2}{x_1} \xi_1 \xi_2, \quad x'_2 = \frac{1 + x_1 + x_2}{x_1 x_2} + \frac{\xi_1 \xi_2}{x_1}, \quad x''_1 = \frac{1 + x_1}{x_2} (1 - \xi_1 \xi_2),$$

and next:

$$x''_2 = x_1 (1 - \xi_1 \xi_2), \quad x'''_1 = x_2 (1 + \xi_1 \xi_2), \quad \dots$$

The Laurent phenomenon is always verified.

Note that, once again, the above process of consecutive mutations does not lead to 5-periodicity. However, once again, the corresponding cluster superalgebra is generated by the following functions $\{x_1, x_2, x'_1, x'_2, x''_1, \xi_1, \xi_2\}$.

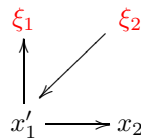
The presymplectic form in this example is:

$$\omega = -\frac{dx_1 \wedge dx_2}{x_1 x_2} + \frac{d(\xi_1 \xi_2) \wedge dx_1}{x_1}.$$

Substituting to this form $x_1 = \frac{1}{x'_1} (1 + x_2(1 + \xi_1 \xi_2))$, one obtains:

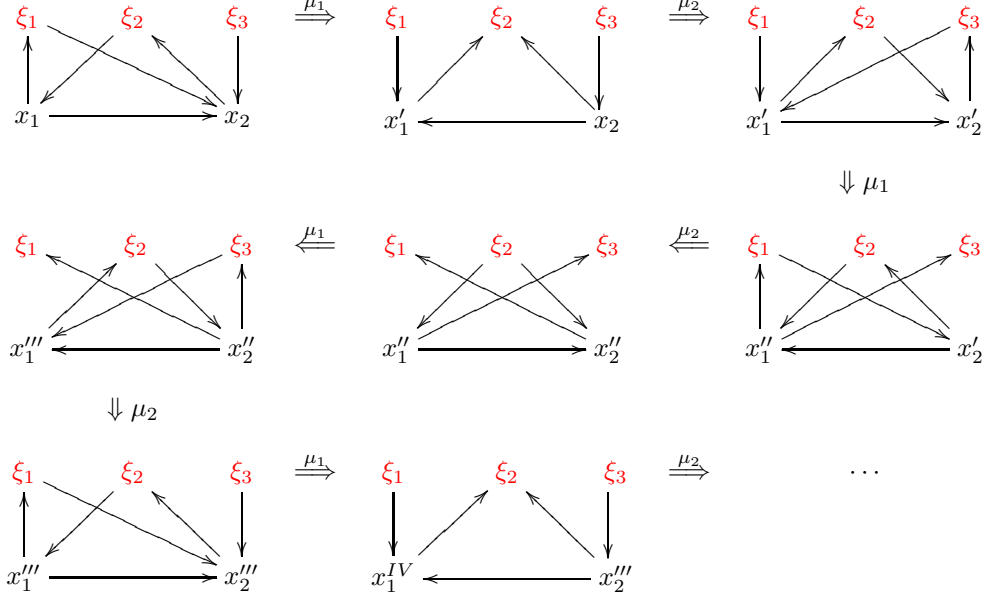
$$\begin{aligned} \omega &= \frac{dx'_1 \wedge dx_2}{x'_1 x_2} - \frac{d(\xi_1 \xi_2) \wedge dx'_1}{x'_1} - \frac{x_2 d(\xi_1 \xi_2) \wedge dx_2}{(1 + x_2 + \xi_1 \xi_2)x_2} + \frac{d(\xi_1 \xi_2) \wedge dx_2}{(1 + x_2 + \xi_1 \xi_2)x_2} \\ &= \frac{dx'_1 \wedge dx_2}{x'_1 x_2} - \frac{d(\xi_1 \xi_2) \wedge dx'_1}{x'_1}. \end{aligned}$$

Since this expression is nothing but the presymplectic form corresponding to the following extended quiver:



We conclude, that the 2-form ω is, again, invariant with respect to the mutation μ_1 . It is not difficult to check that ω is invariant with respect to all the other mutations we consider in this example.

Example 4.5. The following extended quiver with two even and three odd vertices, as well as its mutations, will be particularly important for us (see Section 5 below). As before, let us perform an infinite sequence of mutations:



One then has:

$$\begin{aligned} x'_1 &= \frac{1+x_2}{x_1} - \frac{\xi_1 \xi_2}{x_1}, & x'_2 &= \frac{1+x_1+x_2}{x_1 x_2} - \frac{\xi_1 \xi_2}{x_1 x_2} - \frac{\xi_2 \xi_3}{x_2}, \\ x''_1 &= \frac{1+x_1}{x_2} + \frac{x_1}{x_2} (\xi_1 + \xi_3) \xi_2, & x''_2 &= x_1 (1 + \xi_1 \xi_2) \\ x'''_1 &= x_2 (1 + \xi_2 \xi_3), \end{aligned}$$

and so on. The next mutations of the quiver and of the coordinates can be easily calculated, and once again, the Laurent phenomenon is verified.

The presymplectic form in this example is as follows:

$$\omega = \frac{dx_1 \wedge dx_2}{x_1 x_2} - \frac{d(\xi_1 \xi_2) \wedge dx_1}{x_1} + \frac{d(\xi_1 \xi_2) \wedge dx_2}{x_2} - \frac{d(\xi_2 \xi_3) \wedge dx_2}{x_2}.$$

Although the computation in this case is slightly more involved, one readily checks its invariance under the above mutations.

Example 4.6. The supergroup $\text{OSp}(1|2)$. One of the first examples of cluster algebras given in [14] is the algebra of regular functions on the Lie group $\text{SL}(2)$. We consider here its superanalog.

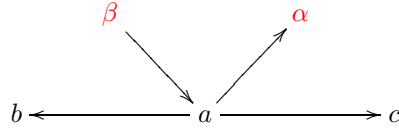
The most elementary superanalog of the group $\text{SL}(2)$ is the supergroup $\text{OSp}(1|2)$. For more details about properties and applications of this supergroup, see [29]. Let $\mathcal{R} = \mathcal{R}_0 \oplus \mathcal{R}_1$ be a commutative ring. The set of \mathcal{R} -points of the supergroup $\text{OSp}(1|2)$ is the following $3|2$ -dimensional

supergroup of matrices:

$$(3) \quad \left(\begin{array}{cc|c} a & b & \gamma \\ c & d & \delta \\ \hline \alpha & \beta & e \end{array} \right) \quad \text{such that} \quad \begin{aligned} ad &= 1 + bc - \alpha\beta, \\ e &= 1 + \alpha\beta, \\ \gamma &= a\beta - b\alpha \\ \delta &= c\beta - d\alpha. \end{aligned}$$

The elements $a, b, c, d, e \in \mathcal{R}_0$, and $\alpha, \beta, \gamma, \delta \in \mathcal{R}_1$; these elements are generators of the algebra of regular functions on $\text{OSp}(1|2)$.

Choose the initial cluster coordinates (a, b, c, α, β) , and consider the following quiver:



The coordinate d is then the mutation of a , i.e., $a' = d$. Indeed, the exchange relation (1) for the coordinate a reads

$$aa' = 1 + bc + \beta\alpha,$$

which is precisely the first equation for $\text{OSp}(1|2)$ relating a and d . Note that, similarly to the SL_2 -case the coordinates b and c are frozen cf. [14].

A challenging problem is to find more geometric examples of of cluster superalgebras: Lie supergroups, Grassmannians, etc.

5. SUPERFRIEZES AND CLUSTER SUPERALGEBRAS OF TYPE A

Frieze patterns were invented by Coxeter [9] (see also [8]). This notion provides surprising relations between classical continued fractions, projective geometry (cross-ratios) and quiver representations. Coxeter’s friezes are also related to linear difference equations and the classical moduli spaces $\mathcal{M}_{0,n}$ of configurations of points, see [35]. The set of Coxeter’s friezes is an algebraic variety that has a structure of cluster algebras, associated to the Dynkin quivers A_n . For a modern survey, see [34].

The notion of *superfrieze* was introduced in [37] as generalization of Coxeter’s frieze patterns. The collection of all superfriezes is an algebraic supervariety isomorphic to the supervariety of supersymmetric Hill’s (or one-dimensional Schrödinger) equations [35] with some particular monodromy condition.

In this section, we describe the structure of cluster superalgebra on the supervariety of superfriezes. The underlying quiver will be A_n , the extended quiver \tilde{Q} is obtained by adding $n + 1$ odd coordinates. We start with the description of the supergroup $\text{OSp}(1|2)$ that is used to develop the theory of superfriezes.

5.1. Supersymmetric discrete Schrödinger equation. Consider two infinite sequences of elements of some supercommutative ring \mathcal{R} :

$$(a_i), \quad (\beta_i), \quad i \in \mathbb{Z},$$

where $a_i \in \mathcal{R}_0$ and $\beta_i \in \mathcal{R}_1$.

The following equation with indeterminate $(V_i, W_i)_{i \in \mathbb{Z}}$:

$$(4) \quad \begin{pmatrix} V_{i-1} \\ V_i \\ W_i \end{pmatrix} = A_i \begin{pmatrix} V_{i-2} \\ V_{i-1} \\ W_{i-1} \end{pmatrix}, \quad \text{where} \quad A_i = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & a_i & -\beta_i \\ \hline 0 & \beta_i & 1 \end{array} \right),$$

is the supersymmetric version of discrete Schrödinger equation, see [37]. Note that the matrix A_i belongs to the supergroup $\text{OSp}(1|2)$.

We assume that the coefficients a_i, β_i are (anti)periodic with some period n :

$$a_{i+n} = a_i, \quad \beta_{i+n} = -\beta_i,$$

for all $i \in \mathbb{Z}$. Under this assumption, there is a notion of *monodromy*, i.e., an element $M \in \text{OSp}(1|2)$, such that periodicity properties of the solutions of (4) are described by M .

Supersymmetric discrete Schrödinger equations with fixed monodromy matrix:

$$(5) \quad M = \left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array} \right).$$

considered in [37]. This is an algebraic supervariety of dimension $n|(n+1)$ which is a version of super moduli space $\mathfrak{M}_{0,n}$, see [41]. The notion of superfrieze allows one to define special coordinates on this supervariety.

5.2. The definition of a superfrieze and the corresponding superalgebra. Similarly to the case of classical Coxeter's friezes, a superfrieze is a horizontally-infinite array bounded by rows of 0's and 1's. Even and odd elements alternate and form "elementary diamonds"; there are twice more odd elements.

Definition 5.1. *A superfrieze, or a supersymmetric frieze pattern, is the following array*

$$\begin{array}{cccccccc}
 & \dots & 0 & & 0 & & 0 & \\
 \dots & 0 & & 0 & & 0 & & 0 & \dots \\
 1 & & & 1 & & 1 & & \dots & \\
 & \varphi_{0,0} & & \varphi_{\frac{1}{2},\frac{1}{2}} & & \varphi_{1,1} & & \varphi_{\frac{3}{2},\frac{3}{2}} & & \varphi_{2,2} & \dots \\
 & & f_{0,0} & & f_{1,1} & & f_{2,2} & & & & \\
 & \varphi_{-\frac{1}{2},\frac{1}{2}} & & \varphi_{0,1} & & \varphi_{\frac{1}{2},\frac{3}{2}} & & \varphi_{1,2} & & \varphi_{\frac{3}{2},\frac{5}{2}} & \dots \\
 f_{-1,0} & & & f_{0,1} & & f_{1,2} & & & & & \\
 & \ddots & & \ddots & & \ddots & & \ddots & & \ddots & \\
 & & f_{2-m,1} & & f_{0,m-1} & & f_{1,m} & & & & \\
 \dots & \varphi_{\frac{3}{2}-m,\frac{3}{2}} & & \varphi_{2-m,2} & & \dots & & \varphi_{0,m} & & \varphi_{\frac{1}{2},m+\frac{1}{2}} & & \varphi_{1,m+1} \\
 1 & & & 1 & & 1 & & & & & \\
 \dots & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
 & \dots & 0 & & 0 & & 0 & & \dots & &
 \end{array}$$

where $f_{i,j} \in \mathcal{R}_0$ and $\varphi_{i,j} \in \mathcal{R}_1$, and where every elementary diamond:

$$\begin{array}{ccc}
 & B & \\
 & \Xi & \Psi \\
 A & & D \\
 & \Phi & \Sigma \\
 & C &
 \end{array}$$

satisfies the following conditions:

$$\begin{aligned}
 (6) \quad AD - BC &= 1 + \Sigma\Xi, \\
 B\Phi - A\Psi &= \Xi, \\
 B\Sigma - D\Xi &= \Psi,
 \end{aligned}$$

that we call the frieze rule.

The integer m , i.e., the number of even rows between the rows of 1's is called the width of the superfrieze.

The last two equations of (6) are equivalent to

$$A\Sigma - C\Xi = \Phi, \quad D\Phi - C\Psi = \Sigma.$$

Note also that these equations also imply $\Xi\Sigma = \Phi\Psi$, so that the first equation can also be written as follows: $AD - BC = 1 - \Xi\Sigma$.

One can associate an elementary diamond with every element of $\text{OSp}(1|2)$ using the following formula:

$$\left(\begin{array}{cc|c} a & b & \gamma \\ c & d & \delta \\ \hline \alpha & \beta & e \end{array} \right) \longleftrightarrow \begin{array}{ccc} & -a & \\ & \gamma & \alpha \\ b & & -c \\ & -\beta & \delta \\ & & d \end{array}$$

so that the relations (3) and (6) coincide.

Consider also the configuration:

$$\begin{array}{cccc}
 & \tilde{\Psi} & \tilde{\Xi} & \\
 & B & & \\
 \tilde{\Phi} & \Xi & \Psi & \tilde{\Sigma} \\
 A & & D & \\
 & \Phi & \Sigma & \\
 & C & &
 \end{array}$$

The frieze rule (6) then implies

$$B(\Phi - \tilde{\Phi}) = A(\Psi - \tilde{\Psi}), \quad B(\Sigma - \tilde{\Sigma}) = D(\Xi - \tilde{\Xi}).$$

Definition 5.2. The supercommutative superalgebra generated by all the entries of a superfrieze will be called the algebra of a superfrieze.

5.3. Examples: superfriezes of width 1 and 2. The most general superfrieze of width $m = 1$ is of the following form:

$$\begin{array}{cccccccc}
 & 0 & & 0 & & 0 & & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & & & 1 & & 1 & & 1 \\
 & \xi & \xi & \xi' & \xi' & \xi - x\eta & \xi - x\eta & \eta & \eta \\
 & x & & x' & & x & & x' \\
 & \xi - x\eta & x\eta - \xi & \eta & -\eta & -\xi & \xi & -\xi' & \xi' \\
 1 & & & 1 & & 1 & & 1 & 1 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & & 0 & & 0 & & 0 &
 \end{array}$$

where

$$x' = \frac{2}{x} + \frac{\eta\xi}{x}, \quad \xi' = \eta - \frac{2\xi}{x}.$$

One can chose local coordinates (x, ξ, η) to parametrize the supervariety of superfriezes. The value of x' precisely corresponds to the mutation of x in Example 4.1.

The next example is a superanalog of so-called Gauss' "Pentagramma mirificum":

$$\begin{array}{cccccccc}
 0 & & 0 & & 0 & & 0 & & 0 & \dots \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 \dots & 1 & & 1 & & 1 & & 1 & & 1 \\
 & \xi^* & \xi & \xi & \xi' & \xi' & \textcircled{1} & \textcircled{1} & \zeta^* & \zeta^* \\
 y' & & x & & x' & & x'' & & y & \\
 & -\eta' & \eta^* & \textcircled{2} & \eta & \textcircled{2}' & \eta' & \eta^* & -\eta & \eta \\
 & x'' & & y & & y' & & x & & x' \\
 & \textcircled{1} & -\textcircled{1} & \zeta^* & -\zeta^* & \zeta & -\zeta & \zeta' & -\zeta' & -\xi' \\
 1 & & 1 & & 1 & & 1 & & 1 & \dots \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 \dots & 0 & & 0 & & 0 & & 0 & & 0
 \end{array}$$

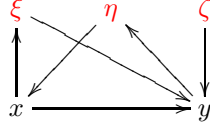
The frieze is defined by the initial values (x, y, ξ, η, ζ) , the next values are easily calculated using the frieze rule:

$$x' = \frac{1+y}{x} + \frac{\eta\xi}{x}, \quad y' = \frac{1+x+y}{xy} + \frac{\eta\xi}{xy} + \frac{\zeta\eta}{y}.$$

Note that these formulas coincide with those of mutations of coordinates x_1, x_2 in Example 4.5. One then calculates:

$$x'' = \frac{1+y'}{x'} + \frac{\eta'\xi'}{x'} = \frac{1+x}{y} + \frac{\eta\xi}{y} + \xi\zeta + \frac{x}{y}\zeta\eta.$$

All these Laurent polynomials can be obtained as mutations of the initial coordinates (x, y, ξ, η, ζ) and the initial quiver



see Example 4.5.

For the odd coordinates, one has:

$$\xi' = \eta - x'\xi = \eta - \frac{1+y}{x}\xi, \quad \eta' = \zeta - y'\xi = \zeta - \frac{1+x+y}{xy}\xi - \frac{\xi\eta\zeta}{y}, \quad \zeta' = -\xi$$

On the other side of the initial diagonal,

$$\zeta^* = \eta - y\zeta, \quad \eta^* = \xi - x\zeta, \quad \xi^* = -\zeta.$$

Furthermore,

$$\textcircled{1} = \frac{(1+x)}{y}\eta - \xi - \zeta, \quad \textcircled{2} = x\eta - y\xi,$$

and finally:

$$\textcircled{2}' = x'\zeta - y'\eta = \frac{1+y}{x}\zeta - \frac{1+x+y}{xy}\eta - \frac{\xi\eta\zeta}{x}.$$

5.4. Properties of superfriezes. The main properties of superfriezes are similar to those of the classical Coxeter friezes, see [37].

(a) The property of *glide symmetry* reads:

$$f_{i,j} = f_{j-m-1,i-2}, \quad \varphi_{i,j} = \varphi_{j-m-\frac{3}{2},i-\frac{3}{2}}, \quad \varphi_{i+\frac{1}{2},j+\frac{1}{2}} = -\varphi_{j-m-1,i-1}.$$

This implies, in particular, the following (anti)periodicity:

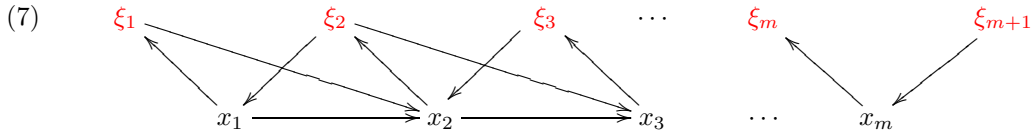
$$\varphi_{i+n,j+n} = -\varphi_{i,j}, \quad f_{i+n,j+n} = f_{i,j},$$

for all $i, j \in \mathbb{Z}$.

(b) The Laurent phenomenon: entries of a superfrieze are Laurent polynomials in the entries from any of its diagonals.

(c) The collection of all superfriezes of width n is an algebraic supervariety of superdimension $n|(n+1)$. It is isomorphic to the supervariety of Schrödinger equations (4) with monodromy condition (5). The relation to difference equations is as follows. The entries of the South-East diagonal of every superfrieze are solutions to the discrete Schrödinger equation (4).

5.5. Superfriezes and cluster superalgebras. Let us now describe the cluster structure of the supervariety of superfriezes. Consider the following quiver with m even and $m+1$ odd vertices:



and the corresponding cluster superalgebra.

Indeed, the frieze rule implies that the entry between ξ_1 and ξ'_1 (previously denoted by $*$) is also equal to ξ'_1 , i.e., we have the following fragment of the superfrieze:

$$\begin{array}{ccccc}
 & & 1 & & 1 \\
 & & \xi_1 & & \xi'_1 & & \xi'_1 \\
 x_1 & & & & & & x'_1 \\
 & & * & & \xi_2 & &
 \end{array}$$

The above expression for ξ'_1 is just the third equality in (6). It follows that ξ'_1 belongs to the cluster superalgebra $A(\tilde{\mathcal{Q}})$.

It was proved in [37] that the entries on the diagonals of the superfrieze satisfy recurrence equations with coefficients standing in the first two rows. In particular, Lemma 2.5.3 of [37] implies the following recurrence for the odd entries of the superfrieze:

$$\xi'_k - \xi'_{k-1} = -\xi_1 x'_k, \quad \text{for all } k.$$

One concludes, by induction on k , that all of the entries $\{\xi'_1, \dots, \xi'_{m+1}\}$ belong to the cluster superalgebra $A(\tilde{\mathcal{Q}})$. Again, using the induction one arrives at the same conclusion for all parallel diagonals.

Finally, one proves in a similar way that the entries in-between, denoted by $*$, also belong to the cluster superalgebra $A(\tilde{\mathcal{Q}})$. \square

6. THE LAURENT PHENOMENON AND INVARIANCE OF THE PRESYMPLECTIC FORM

In this section, we prove Theorems 1 and 2.

6.1. Proof of the Laurent phenomenon. Our proof goes along the same lines as the proof of Theorem 3.2 of [14] (see also Theorem 2.1 of [15]). We will repeat this proof giving the additional arguments when necessary.

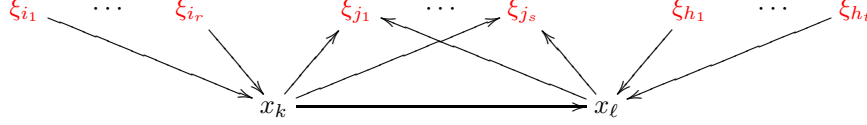
Lemma 6.1. *Given an extended quiver $\tilde{\mathcal{Q}}$ such that all the 2-paths in $\tilde{\mathcal{Q}}_1$ are without multiplicities, the mutation at a given vertex x_k is allowed if and only if for every vertex $x_\ell \in \mathcal{Q}$ connected to x_k by an outgoing arrow $x_k \rightarrow x_\ell$, (at least) one of the following conditions is satisfied:*

- (a) $I_k = I_\ell$;
- (b) $J_k = J_\ell$;
- (c) $I_k = J_k = \emptyset$.
- (d) $I_k = J_\ell$, and $J_k = I_\ell$;
- (e) $I_\ell = J_\ell = \emptyset$.

Proof. It is obvious that any of the conditions (a)-(e) is sufficient. Indeed, the mutation at x_k sends all 2-paths $(\xi_i \rightarrow x_k \rightarrow \xi_j)$ to every vertex $x_\ell \in \mathcal{Q}$ connected to x_k by an outgoing arrow $(x_k \rightarrow x_\ell)$, creating 2-paths $(\xi_i \rightarrow x_\ell \rightarrow \xi_j)$. In cases (a)-(c), at least one of the sets I_ℓ or J_ℓ remains unchanged; in case (d), all arrows connecting x_ℓ to odd vertices disappear; in case (e), this is also clear.

Conversely, assume that $I_k \neq I_\ell$ and $J_k \neq J_\ell$, and the sets are non-empty. Choose $i \in I_k$ and $i \notin I_\ell$, and $j \notin J_k$ and $j \in J_\ell$, then ξ_i is not connected to ξ_j by a 2-path $(\xi_i \rightarrow x_\ell \rightarrow \xi_j)$ this is a contradiction. \square

Example 6.2. The following example illustrates the case (b) of the above lemma.



The mutation of this quiver at x_k is allowed.

Given an initial seed $(\tilde{Q}^0, \{x^0, \xi^0\})$, and consider a sequence of ℓ consecutive mutations

$$\cdots \circ \mu_k \circ \mu_j \circ \mu_i : (\tilde{Q}^0, \{x^0, \xi^0\}) \mapsto (\tilde{Q}^1, \{x^1, \xi^1\}) \mapsto \cdots \mapsto (\tilde{Q}^\ell, \{x^\ell, \xi^\ell\}).$$

We need to prove that the coordinates $\{x^\ell, \xi^\ell\}$ are Laurent polynomials in $\{x^0, \xi^0\}$. For this end, we will use the induction on $\ell \geq 3$.

Following [14, 15], let us consider the following three seeds:

$$(\tilde{Q}^1, \{x^1, \xi^1\}) = \mu_i(\tilde{Q}^0, \{x^0, \xi^0\}), \quad (\tilde{Q}^2, \{x^2, \xi^2\}) = \mu_j \circ \mu_i(\tilde{Q}^0, \{x^0, \xi^0\}),$$

and

$$(\tilde{Q}^3, \{x^3, \xi^3\}) := \mu_i \circ \mu_j \circ \mu_i(\tilde{Q}^0, \{x^0, \xi^0\}).$$

Using the inductive assumption, we suppose that $\{x^\ell, \xi^\ell\}$ are Laurent polynomials in $\{x^1, \xi^1\}$ and $\{x^2, \xi^2\}$, since these seeds are closer to $\{x^\ell, \xi^\ell\}$ in the sequence of mutations. Furthermore, $\{x^\ell, \xi^\ell\}$ must also be Laurent polynomials in $\{x^3, \xi^3\}$, since applying μ_i to $(\tilde{Q}^3, \{x^3, \xi^3\})$, one obtains the seed $\mu_i^2 \circ \mu_j \circ \mu_i(\tilde{Q}^0, \{x^0, \xi^0\})$ that generates the same cluster algebra as $(\tilde{Q}^2, \{x^2, \xi^2\})$ (cf. Remark 3.2).

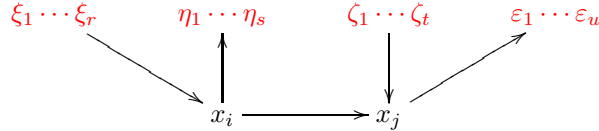
We now need the following analog of Lemma 3.3 of [14] (see also Lemma 2.2 of [15]).

Lemma 6.3. (i) The coordinates $\{x^1, \xi^1\}$, $\{x^2, \xi^2\}$ and $\{x^3, \xi^3\}$ are Laurent polynomials in $\{x^0, \xi^0\}$.

(ii) Furthermore, $\gcd(x_i^3, x_i^1) = \gcd(x_j^2, x_i^1) = 1$, where the greatest common divisor is taken over the ring of polynomials in $\{x_1, \dots, x_n\}$.

Proof. Part (i). The coordinates $\{x^1, \xi^1\}$, $\{x^2, \xi^2\}$ are obviously Laurent polynomials in $\{x^0, \xi^0\}$. In the case where the vertices x_i and x_j are not connected by an arrow, the mutations μ_i and μ_j commute, and it is again obvious that the coordinates $\{x^3, \xi^3\}$ are Laurent polynomials in $\{x^0, \xi^0\}$. Let us calculate the mutation $\mu_i \circ \mu_j \circ \mu_i$ in the case where the vertices x_i and x_j are connected by an arrow. We assume that the arrow is oriented as follows $x_i \rightarrow x_j$ (the computations in case $x_i \leftarrow x_j$ are similar).

Let us start with an arbitrary extended quiver a fragment of which is as follows:



and show that the Laurent phenomenon occurs if and only if the mutation is allowed. One obtains from (1):

$$x_i^1 = \frac{M_1^i + x_j M_2^i}{x_i} + \frac{(\sum \xi)(\sum \eta) M_1^i}{x_i},$$

where M_1^i and $x_j M_2^i$ are the monomials obtained by the product of the even coordinates connected to x_i by ingoing and outgoing arrows, respectively. Similarly,

$$x_j^2 = \frac{M_1^i M_1^j + x_i^1 M_2^j}{x_j} + \frac{(\sum \zeta)(\sum \varepsilon) M_1^j}{x_j}.$$

The coordinates x_i^1 and x_j^2 are, of course, trivially Laurent polynomials in the initial coordinates. Next, after a (quite long) computation, one obtains:

$$x_i^3 = \frac{M_2^j + x_i M_1^j}{x_j} + \frac{(\sum \zeta)(\sum \varepsilon) M_1^j x_i}{x_j} - \frac{(\sum \xi)(\sum \eta)(\sum \zeta)(\sum \varepsilon) M_1^i M_1^j x_i}{x_j (M_1^i + x_j M_2^i)}.$$

(Note that the coefficient of the term $(\sum \xi)(\sum \eta)$ vanishes identically.) We conclude that x_i^3 is a Laurent polynomial if and only if the product $(\sum \xi)(\sum \eta)(\sum \zeta)(\sum \varepsilon)$ vanishes. This is guaranteed by (and almost equivalent to) the assumption that the mutation μ_1 is allowed.

Indeed, the following statement provides with a necessary and sufficient condition for the mutation at a vertex x_k to be allowed.

The second part of the lemma stating that x_i^1 and x_i^3 , and x_i^1 and x_j^2 are coprime follows from the similar result in the poorly even case. Indeed, our expressions for x_i^1, x_i^3 and x_j^2 differ from their purely even projections by nilpotent terms. Therefore, these expressions are coprime if and only if their purely even parts are coprime.

Hence the Lemma. \square

The rest of the proof is identical to those of Theorem 3.2 from [14] (and of Theorem 2.1 from [15]). By inductive assumption, all of the coordinates $\{x_1^\ell, \dots, x_n^\ell, \xi_1^\ell, \dots, \xi_m^\ell\}$ are Laurent polynomials in $\{x_1^0, \dots, x_i^1, \dots, x_n^0, \xi_1, \dots, \xi_m\}$ and also in $\{x_1^0, \dots, x_i^3, \dots, x_j^2, \dots, x_n^0, \xi_1, \dots, \xi_m\}$. But the fact that the mutated coordinates are themselves Laurent polynomials and coprime implies that the coordinates $\{x_1^\ell, \dots, x_n^\ell, \xi_1^\ell, \dots, \xi_m^\ell\}$ are, indeed, Laurent polynomials in the initial coordinates.

Theorem 1 is proved.

6.2. The presymplectic form is invariant. We prove Theorem 2 stating that, for every extended quiver $\tilde{\mathcal{Q}}$, the form ω is invariant under mutations at even coordinates. Our proof goes along the same lines as the elementary proof in the classical case, presented in Section 1.2.

Let us choose an even vertex $x_k \in \tilde{\mathcal{Q}}$. Perform a mutation at x_k , and denote the 2-form associated to the quiver after the mutation by ω' . Our goal is to check that $\omega = \omega'$.

As above, in the purely even case, collect the terms in ω containing x_k ; one obtains:

$$\omega_{x_k} := \frac{dx_k \wedge d \frac{x_{J_k}}{x_{I_k}}}{x \frac{x_{J_k}}{x_{I_k}}} + \sum_{\xi_i \rightarrow x_k \rightarrow \xi_j} \frac{d(\xi_i \xi_j) \wedge dx_k}{x_k},$$

where, as above, the monomial $x_{I_k} = x_{i_1} \cdots x_{i_r}$ is the product of all even coordinates connected to x_k by the ingoing arrows ($x_i \rightarrow x_k$), and where $x_{J_k} = x_{j_1} \cdots x_{j_s}$ is the product of all even coordinates connected to x_k by the outgoing arrows ($x_k \rightarrow x_j$).

The exchange relation (1) can be rewritten as follows:

$$x_k = \frac{1}{\widetilde{x_k'}} \left(1 + \frac{x_{J_k}}{x_{I_k}} + \sum_{\xi_i \rightarrow x_k \rightarrow \xi_j} \xi_i \xi_j \right),$$

where $\widetilde{x_k'} := \frac{x_k'}{x_{I_k}}$. Substituting this expression into ω_{x_k} , and applying the Leibniz rule, one obtains

$$\frac{dx_k \wedge d \frac{x_{J_k}}{x_{I_k}}}{x_k \frac{x_{J_k}}{x_{I_k}}} = - \frac{\widetilde{x_k'} \wedge d \frac{x_{J_k}}{x_{I_k}}}{\widetilde{x_k'} \frac{x_{J_k}}{x_{I_k}}} + \sum_{\xi_i \rightarrow x_k \rightarrow \xi_j} \frac{d(\xi_i \xi_j) \wedge d \frac{x_{J_k}}{x_{I_k}}}{\left(1 + \frac{x_{J_k}}{x_{I_k}} \right) \frac{x_{J_k}}{x_{I_k}}},$$

for the first term, and

$$\sum_{\xi_i \rightarrow x_k \rightarrow \xi_j} \frac{d(\xi_i \xi_j) \wedge dx_k}{x_k} = - \sum_{\xi_i \rightarrow x_k \rightarrow \xi_j} \frac{d(\xi_i \xi_j) \wedge d\tilde{x}'_k}{\tilde{x}'_k} + \sum_{\xi_i \rightarrow x_k \rightarrow \xi_j} \frac{d(\xi_i \xi_j) \wedge d\frac{x_{J_k}}{x_{I_k}}}{1 + \frac{x_{J_k}}{x_{I_k}}}$$

for the second term. Collecting the above two terms, one finally gets:

$$\omega_{x_k} = - \frac{dx'_k \wedge d\frac{x_{J_k}}{x_{I_k}}}{x'_k \frac{x_{J_k}}{x_{I_k}}} - \sum_{\xi_i \rightarrow x_k \rightarrow \xi_j} \frac{d(\xi_i \xi_j) \wedge dx'_k}{x'_k} + \frac{dx_{I_k} \wedge dx_{J_k}}{x_{I_k} x_{J_k}} + \sum_{\xi_i \rightarrow x_k \rightarrow \xi_j} \frac{d(\xi_i \xi_j) \wedge dx_{J_k}}{x_{J_k}}.$$

We need to prove that this expression coincides with the term $\omega'_{x'_k}$ in the 2-form ω' . Indeed, the first two terms with minus signs correspond to the reversing of arrows at x'_k ; the third term is the additional term generated by the rule (1) of the mutation of the classical quiver \mathcal{Q} ; the last term is generated by the step (1*) of the mutation of the extended quiver $\tilde{\mathcal{Q}}$.

We have proved that $\omega_{x_k} = \omega'_{x'_k}$, and hence $\omega = \omega'$, since all the other terms of the 2-form remain unchanged under the mutation at x_k .

Theorem 2 is proved.

7. AN APPLICATION: EXTENDED SOMOS-4 SEQUENCE

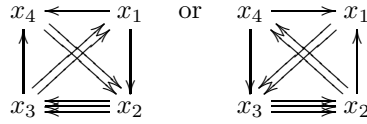
The goal of this section is to show that cluster superalgebras can have very concrete applications. We present two sequences of numbers constructed as extensions of the well-known *Somos-4 sequence*. We deduce from the Laurent phenomenon for superalgebras that all entries of these sequences are integers.

Recall that the Somos-4 sequence is the sequence of numbers $(a_n)_{n \in \mathbb{N}}$ defined by the recurrence

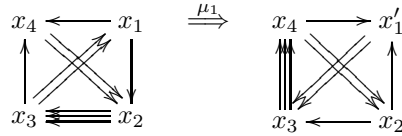
$$a_{n+4} = \frac{a_{n+1}a_{n+3} + a_{n+2}^2}{a_n},$$

and the initial conditions $a_1 = a_2 = a_3 = a_4 = 1$, and it turns out that the sequence is integer. The first values in the sequence are: 1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, 8209, 83313, 620297, 7869898 ...

A conceptual proof of integrality, obtained in [15], is an application of the Laurent phenomenon. Indeed, consider one of the following quiver (that differ only by orientation):



Each of them performs a cyclic rotation under the mutation at x_1 :



that follows immediately from the definition of quiver mutations (see Section 1.1). The exchange relation reads: $x_1 x'_1 = x_2 x_4 + x_3^2$. Choosing $x_5 := x'_1$, and performing an infinite cyclic process of

mutations: $x_6 = x'_2, x_7 = x'_3, \dots$, one obtains a sequence of rational functions x_n in the initial variables x_1, x_2, x_3, x_4 , satisfying the Somos recurrence. By Laurent phenomenon, every function x_n is a Laurent polynomial. Therefore, choosing the initial values $x_1 = x_2 = x_3 = x_4 = 1$, one obtains an integer sequence.

More interesting examples of cyclically invariant quivers and corresponding sequences can be found in [17], and of sequences with the Laurent property in [15] and [24].

Our extension of the Somos-4 sequence is as follows. Consider a pair of sequences of numbers, $(a_n), (b_n)_{n \in \mathbb{N}}$, that we write in the form of a linear function in a formal variable ε :

$$A_n = a_n + b_n \varepsilon,$$

such that $\varepsilon^2 = 0$. This expression is known under the name of “dual numbers” (invented by Clifford and used by Study, Grassmann, Blaschke,... for geometric purposes). The following statement is a corollary of Theorem 1.

Corollary 7.1. *Suppose that the sequence (A_n) satisfies the recurrence:*

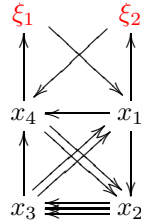
$$(9) \quad A_{n+4} = \frac{A_{n+1}A_{n+3} + A_{n+2}^2(1 + \varepsilon)}{A_n},$$

with the initial conditions:

$$A_1 = A_2 = A_3 = A_4 = 1,$$

i.e., with $a_1 = a_2 = a_3 = a_4 = 1$ and $b_1 = b_2 = b_3 = b_4 = 0$. Then both sequences, (a_n) and (b_n) , are integer.

Proof. The following extended quiver:



performs a cyclic rotation under the mutation at x_1 , cf. Definition 2.3. The exchange relation (1) reads $x_1 x'_1 = x_2 x_4 + (1 + \xi_1 \xi_2) x_3^2$. Setting $\varepsilon = \xi_1 \xi_2$, and applying an infinite cyclic sequence of mutations, one obtains a sequence of rational functions $x_n(x_1, x_2, x_3, x_4, \xi_1, \xi_2)$ satisfying the recursion (9). The result then follows from the Laurent phenomenon. \square

The sequence (a_n) is obviously just the Somos-4 sequence. The first values of the sequence (b_n) are: 0, 0, 0, 0, 1, 2, 10, 48, 160, 1273, 7346, 51394, 645078, 5477318, 87284761 . . . This sequence cannot be found in OEIS.

Remark 7.2. Rewriting (9) componentwise, one gets the following, more explicit, but less pleasant recurrence for the sequence (b_n) :

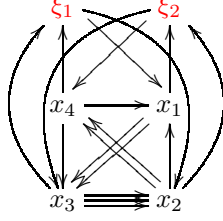
$$b_{n+4} = \frac{a_{n+1}b_{n+3} + 2a_{n+2}b_{n+2} + a_{n+3}b_{n+1} + a_{n+2}^2}{a_n} - \frac{(a_{n+3}a_{n+1} + a_{n+2}^2) b_n}{a_n^2}.$$

Yet another relation which is also equivalent to the above recurrence is:

$$b_{n+4}a_n + b_n a_{n+4} = a_{n+1}b_{n+3} + 2a_{n+2}b_{n+2} + a_{n+3}b_{n+1} + a_{n+2}^2$$

(communicated to me by Michael Somos).

Consider, furthermore, the following extended quiver:



It is not difficult to see that under the cyclic sequence of mutations, $\mu_1, \mu_2, \mu_3, \mu_4, \mu_1, \dots$ this quiver also rotates cyclically, and the exchange relation (1) reads this time: $x_1 x'_1 = x_3^2 + (1 + \xi_1 \xi_2) x_2 x_4$. One then obtains another extension, $\tilde{A}_n = a_n + \tilde{b}_n \varepsilon$, of the Somos-4 sequence, satisfying the recursion

$$\tilde{A}_{n+4} = \frac{\tilde{A}_{n+2}^2 + \tilde{A}_{n+1} \tilde{A}_{n+3} (1 + \varepsilon)}{\tilde{A}_n}.$$

It follows from Theorem 1, that the sequence $(\tilde{b}_n)_{n \in \mathbb{N}}$ is, again, integer. The first values of this sequence are: 0, 0, 0, 0, 1, 3, 10, 59, 198, 1387, 9389, 57983, 752301, 6851887, 97297759...

8. VARIATIONS ON FIBONACCI: EXTENDED KRONECKER QUIVER

In this section, we give another, more elementary illustration of the Laurent phenomenon of cluster superalgebras.

The classical Fibonacci sequence $(F_n) = 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \dots$ can also be understood within the cluster algebra framework. Consider the sequence (a_n) , satisfying the recurrence

$$a_{n+2} = \frac{a_{n+1}^2 + 1}{a_n},$$

and the initial conditions $a_0 = a_1 = 1$. The sequence $(a_n)_{n \in \mathbb{N}}$ is then the bisected Fibonacci sequence: $a_n = F_{2n-1}$. The above recurrence is known as Cassini's identity.

Our methods lead to a family of extensions of the sequence (a_n) . As before, we consider another sequence (b_n) , coupled with (a_n) in the form of a sequence of dual numbers: $A_n = a_n + b_n \varepsilon$, where $\varepsilon^2 = 0$. One of these extensions is defined by the recurrence

$$(10) \quad A_{n+2} A_n = A_{n+1}^2 \left(1 + (-1)^{\frac{(n+1)(n+2)}{2}} \varepsilon \right) + 1,$$

and the initial conditions $A_0 = A_1 = 1$. Equivalently,

$$b_{n+2} a_n + b_n a_{n+2} = 2b_{n+1} a_{n+1} + (-1)^{\frac{(n+1)(n+2)}{2}} a_{n+1}^2,$$

with $b_0 = b_1 = 0$.

It turns out that the sequence $(b_n)_{n \in \mathbb{N}}$ then also consists of Fibonacci numbers, but this time with even indices, and taken in a surprising order:

n	0	1	2	3	4	5	6	7	...
a_n	1	1	2	5	13	34	89	233	...
b_n	0	0	1	8	21	21	55	377	...

More precisely,

$$b_n = \begin{cases} F_{2n}, & n \equiv 0, 3 \pmod{4}, \\ F_{2n-2}, & n \equiv 1, 2 \pmod{4}. \end{cases}$$

Our next goal is to explain the origin of this sequence and its relation to cluster superalgebra.

The classical sequence (a_n) can be generated by consecutive mutations $\mu_0, \mu_1, \mu_0, \mu_1, \dots$ of the quiver with two vertices and two arrows:

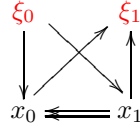
$$x_0 \rightleftarrows x_1,$$

called the 2-Kronecker quiver³. Indeed, the quiver mutation is just the orientation reversing:

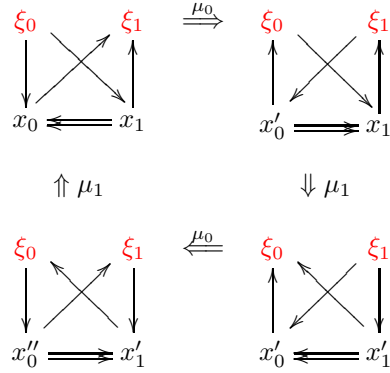
$$x_0 \rightleftarrows x_1 \xrightarrow{\mu_0} x'_0 \rightrightarrows x_1,$$

while the exchange relations read: $x_0 x'_0 = x_1^2 + 1$.

Let us consider the following extended Kronecker quiver:



The sequence of consecutive mutations of this extended Kronecker quiver is 4-periodic:



The exchange relations are:

$$x_0^{(n+1)} x_0^{(n)} = x_1^{(n)2} (1 + (-1)^n \xi_0 \xi_1) + 1, \quad x_1^{(n+1)} x_1^{(n)} = x_0^{(n+1)2} (1 + (-1)^n \xi_0 \xi_1) + 1,$$

which corresponds to the recurrence (10) with $\varepsilon = \xi_0 \xi_1$.

More generally, starting with the extended Kronecker quiver with two odd variables related by an arbitrary number k 2-path through x_0 , and ℓ 2-path through x_1 , one obtains a two-parameter family of integer sequences extending (a_n) :

n	0	1	2	3	4	5	6	7	...
a_n	1	1	2	5	13	34	89	233	...
b_n	0	0	k	$4(\ell + k)$	$20\ell + k$	$43\ell - 22k$	$88\ell - 33k$	$399\ell - 22k$...

The previously considered case of (b_n) given by Fibonacci numbers, corresponds to $k = \ell = 1$.

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³ Note that the Fibonacci sequence is also related to the 3-Kronecker quiver, see [11].

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