Theorem on six vertices of a plane curve via the Sturm theory

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Abstract

We discuss the theorem on the existence of six points on a convex closed plane curve in which the curve has a contact of order six with the osculating conic. (This is the "projective version" of the well known four vertices theorem for a curve in the Euclidean plane.) We obtain this classical fact as a corollary of some general Sturm-type theorems.

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1 introduction

The well known classical theorem states that a convex curve on the Euclidean plane has at least four vertices (critical points of its curvature). This theorem has been frequently discussed in mathematical literature (see [1, 8]). Beautiful applications of this theorem to symplectic geometry were discovered by V.I. Arnol'd [1, 2, 3]. The relation to the Sturm theory is given by S. Tabachnikov [8]. His proof of the four vertices theorem is based on the fact that a function on S^1 without n first harmonics of the Fourier decomposition vanishes at least $2n$ times.

A point on a locally convex plane curve c is called *sextactic* if the oscullating conic has a contact of order ≥ 6 with c in this point. (Recall, that in a generic point the contact is of order 5 since a conic is defined by 5 points.) Sextactic points can be defined also as critical points of the affine curvature or by the fact that the projective length element of curve c vanishes in these points.

Sextactic points are invariant under projective transformations. This kind of singular points is an analogue of vertices in projective (or affine) geometry. (Recall that in the Euclidean case the osculating circle has a contact of order ≥ 4 with the curve in any vertex.)

The following classical theorem can be considered as the "projective analogue" of the four vertices theorem.

Six vertices theorem. *A closed convex curve on $\mathbf{R}^2$ has at least six sextactic points.*

Corollary. *The affine curvature of a convex closed curve on $\mathbf{R}^2$ has at least six critical points.*

The proof can be found in [5].

The main result of this paper is a series of general Sturm-type theorems (in the spirit of Tabachnikov theorem). We estimate the number of zero points of a function on S^1 orthogonal to solutions of a disconjugate linear differential equation. This approach contains at the same time the four vertices theorem of Euclidean geometry and the six vertices theorem.

2 The Sturm theorems

Consider a linear differential equation on S^1:

$$A\phi(x) = \phi^{(n)}(x) + u_{n-1}(x)\phi^{(n-1)}(x) + \cdots + u_0(x)\phi(x) = 0 \qquad (1)$$

Here $u_i(x) \in C^\infty(S^1)$ (this means, all the potentials u_i are periodic: $u_i(x+1) = u_i(x)$).

Definition. Equation (1) is called *disconjugate on* S^1 if:

1. Order $n = 2k+1$: all the solutions are periodic: $\phi(x+1) = \phi(x)$ and have at most $2k$ zeros (with multiplicity) on S^1.

2. Order $n = 2k$ all the solutions are anti-periodic: $\phi(x+1) = -\phi(x)$ and have at most $2k-1$ zeros (with multiplicity) on S^1.

In these cases, A is called a *disconjugate operator*.

2.1. Theorem 1. *Given a function $f \in C^\infty(S^1)$ orthogonal to all the solutions of a $2n+1$-order disconjugate equation:*

$$\int_{S^1} f(x)\phi(x)dx = 0$$

then f has at least $2n+2$ distinct zero points on S^1.

This is a generalization of the Tabachnikov theorem [8] stating the same fact for a function $f(x) \in C^\infty(S^1)$ without n first harmonics. Indeed, such a function is orthogonal to the solutions of the equation $\partial_x(\partial_x^2+1)(\partial_x^2+4)\cdots(\partial_x^2+n^2)\phi = 0$.

Corollary. A function f in the image of a $2n+1$-order disconjugate operator A ($f = Ag$ where $g \in C^\infty(S^1)$ is any function) vanishes at least $2n+2$ times on S^1.

Indeed, f is orthogonal to the solutions of the equation $A^*\phi = 0$, where A^* is the operator adjoint to A. It is sufficient to remark that the operator A^* is disconjugate if A is disconjugate.

Theorem 2. *Given a function $f \in C^\infty(S^1)$ orthogonal to all the products of solutions of a n-order disconjugate equation:*

$$\int_{S^1} f(x)\phi_1(x)\phi_2(x)dx = 0$$

then f has et least $2n$ distinct zero points on S^1.

Remark. There exist straightforward generalizations of Theorems 1 and 2. It is sufficient to consider a function orthogonal to a product of any 3, 4 etc. solutions of a disconjugate equation.

2.2. Proof of Theorem 1. Consider a function $f \in C^\infty(S^1)$ orthogonal to all the solutions of a disconjugate equation $A\phi = 0$.

First, observe that f has at least one zero. Indeed, there exists a solution ϕ positive almost everywhere on S^1 (take for example a solution vanishing in some point with order $2n$, then the disconjugacy condition implies that it has no more zero points). From $\int_{S^1} f(x)\phi(x)dx = 0$ one concludes that function f changes its sign at least once.

Let us prove that the number of points of S^1 in which function f has odd-order zeros (changes its sign) is superior to $2n$. Suppose that f has $2k$ odd-order zero points $x_1, \dots, x_{2k}$ on S^1 and $k \le n$. Consider a solution ϕ with two properties:

a) ϕ has a zero of order $2(n-k)+1$ in x_1,

b) ϕ vanishes in all points $x_1, \dots x_{2k}$.

The existence of such a solution is evident. In fact, there exists a $2k-1$-dimensional space of solutions vanishing with order $2(n-k)+1$ in x_1. The subspace of this space which consists of solutions vanishing in x_2 has the dimension $\ge 2k-2$, etc. Now, the disconjugacy condition implies that

a) Points $x_1, \dots x_{2k}$ are simple zeros of ϕ,

b) ϕ has no more zeros on S^1.

Finally, (replacing if necessary ϕ by $-\phi$) one obtains that functions $f(x)$ and $\phi(x)$ have the same sign sequence on the segments $]x_1, x_2[,]x_2, x_3[, \dots]x_{2k}, x_1[$ which implies the contradiction:
$\int_{S^1} f(x)\phi(x)dx > 0$. The theorem is proven.

2.3. Proof of Theorem 2 is analogue to those of Theorem 1. Suppose that f has $2k$ odd-order zero points $x_1, \dots, x_{2k}$ on S^1 and $k \le n-1$. Take any number s which is even if n is odd and odd if n is even, such that $k \le s \le n-1$. Then there exists a solution ϕ_1 having odd order zero points in $x_1, \dots, x_s$ and such that it has no more zero points on S^1 (see above). In the same way, there exists a solution ϕ_2 having odd order zero points in $x_{s+1}, \dots, x_{2k}$ and such that it has no more zero points on S^1. Their product $\phi_1\phi_2$ has the same sign sequence as f on the segments $]x_1, x_2[,]x_2, x_3[, \dots]x_{2k}, x_1[$ which implies the contradiction:
$\int_{S^1} f(x)\phi_1\phi_2(x)dx \neq 0$. The theorem is proven.

3 Affine and projective lengths; affine and projective curvatures

We recall some classical definitions of affine and projective geometry of curves. It is very interesting to compare the notion of length in the Euclidean, affine and projective cases. If in the Euclidean case it measures in some sense the distance between the curve and a fixed point, then the affine and the projective lengths measure respectively: the distance between the curve and a straight line, and the distance between the curve and a conic.

3.1. Affine length. Consider a parametrised *locally convex* curve $c(x) = (c_1(x), c_2(x))$ in $\mathbf{R}^2$ (a curve without inflection points). For any x, vectors $c'(x)$ and $c''(x)$ are linearly independent. Define the element of affine length by

$$d\sigma = \begin{vmatrix} c_1'(x) & c_2'(x) \\ c_1''(x) & c_2''(x) \end{vmatrix}^{\frac{1}{3}} dx$$

Then σ is called the *affine parameter*.

3.2. Affine curvature. Vector $c'''(x)$ is a linear combination of $c'(x)$ and $c''(x)$: $c'''(x) = a(x)c''(x) + b(x)c'(x)$. Moreover, the affine parameter σ is characterized by the fact that $c'''(\sigma)$ is collinear to c':

$$c'''(\sigma) = k(\sigma)c'(\sigma) \tag{2}$$

Function $k(\sigma)$ is called the *affine curvature*.

3.3. Wilczynski-Cartan construction [6], [9] (see also [7]).
(i) A parametrised locally convex curve $c(x) \in \mathbf{RP}^2$ canonically defines a linear differential equation of the form:

$$\phi'''(x) = \kappa(x)\phi'(x) + v(x)\phi(x) \tag{3}$$

(ii) Any equation (3) uniquely defines a locally convex curve $c(x) \subset \mathbf{RP}^2$ (modulo projective transformations of $\mathbf{RP}^2$).

Proof. To associate a locally convex curve with an equation (3), consider space E of solutions of (3). Let $V_x \subset E$ consists of solutions vanishing at the moment x. One has a family of 2-dimensional subspaces in a 3-dimensional linear space, or in other words, a curve in

$\mathbf{RP}^2$. It is locally convex (which is easy to verify). In homogeneous coordinates, $c = (\phi_1(x) : \phi_2(x) : \phi_3(x))$ where $\phi_1(x), \phi_2(x), \phi_3(x)$ are any linearly independent solutions of (3). Therefore, the equation (3) is uniquely defined by the corresponding curve.

Lemma 1. *The equation (3) corresponding to a closed convex curve is disconjugate.*

Proof. Consider a closed convex curve $c \subset \mathbf{RP}^2$ (see fig.1). Such a curve has at most two points of intersection with any projective line $\mathbf{RP}^1 \subset \mathbf{RP}^2$. In homogeneous coordinates $c = (\phi_1(x) : \phi_2(x) : \phi_3(x))$ where $\phi_1(x), \phi_2(x), \phi_3(x)$ are solutions of the corresponding equation (3) (see Sec. 2.3). Therefore, any solution of (3) is periodic and has at most 2 zeros on S^1.

3.4. Projective length. Rewrite (3) in more symmetric form:

$$\phi'''(x) = \frac{1}{2}[\kappa(x)\phi'(x) + (\kappa(x)\phi(x))'] + h(x)\phi(x) \tag{4}$$

where $h(x) = v(x) - \kappa'(x)/2$. Remark here that the operator $A_0 = \partial_x^3 - \frac{1}{2}(\kappa(x)\partial_x + \partial_x \kappa(x))$ is antisymmetric.

Definition [6]. *The 1-form on c* $d\sigma = h(x)^{\frac{1}{3}} dx$ *is called the projective length element.*

Remark. The quantity $h(x)$ transforms as a cubic differential $h(x)(dx)^3$ by coordinate transformations. Therefore, the 1-form $d\sigma$ is well defined (see [7]).

The projective length shows how much the curve differs from a conic.

Lemma 2. *c is a conic if and only if* $h \equiv 0$.

Proof. Consider a second order equation

$$\psi''(x) = \frac{\kappa(x)}{4}\psi(x)$$

Verify that the solutions of the equation $A_0\phi = 0$ are given by quadratic polynomials in its solutions. In particular, $\phi_1 = \psi_1^2, \phi_2 = \psi_1\psi_2, \phi_3 = \psi_2^2$ (where ψ_1, ψ_2 are linearly independent) is a basis of solutions. Thus, $\phi_2^2 = \phi_1\phi_3$ and the curve $c = (\phi_1(x) : \phi_2(x) : \phi_3(x))$ is a conic.

3.5. Projective curvature. Let us suppose that $d\sigma \neq 0$ and so σ defines a local parameter on c. Then, the function $\kappa(\sigma)/4$ is called the *projective curvature* of the curve $c(x)$.

3.6. An affine curve as a projective curve. Consider a standard embedding $\mathbf{R}^2 \hookrightarrow \mathbf{RP}^2$ preserving the projective structure on $\mathbf{R}^2$ (see fig.2). An affine locally convex curve $c \subset \mathbf{R}^2$ is embedded to $\mathbf{RP}^2$ as a projective locally convex curve. To define its projective length and projective curvature, represent the equation (2) in the form (4):

$$c'''(\sigma) = \frac{1}{2}[k(\sigma)c'(\sigma) + (k(\sigma)c(\sigma))'] - \frac{1}{2}k'(\sigma)c(\sigma)$$

Therefore, the projective length of c can be defined by the relation:

$$h(\sigma) = -\frac{1}{2}k'(\sigma).$$

On the other hand, any projective curve can be considered (locally) as an affine curve. The equation (3) reduces to the form (2) by a changing of the parameter.

4 Sextactic points

Definition. *A point of a locally convex curve $c \subset \mathbf{RP}^2$ is called sextactic if there exists a conic in $\mathbf{RP}^2$ which has a contact of order ≥ 6 with c in this point.*

4.1. Critical points of the projective length. The notion of a sextactic point can be expressed in terms of the curvature (in affine case) and in terms of the length element (in projective case).

Proposition 1. *A point of a locally convex affine curve $c \subset \mathbf{R}^2$ is sextactic if and only if it is a critical point of the affine curvature.*

Corollary. *A point of a locally convex curve $c \subset \mathbf{RP}^2$ is sextactic if and only if the projective length element $d\sigma$ vanishes at this point.*

Remark here that this statement is just an infinitesimal version of Lemma 2.

Proof of the proposition. Consider a locally convex curve $c \subset \mathbf{RP}^2$. Take the affine parameter on c, then the coordinates of

c satisfies the equation (2). In the neighborhood of point $c_0 = c(0)$ curve c is given by the Taylor series:

$$c(\sigma) = \sigma c_0' + \frac{\sigma^2}{2} c_0'' + \frac{\sigma^3}{6} c_0''' + \frac{\sigma^4}{24} c_0'^{\vee} + \frac{\sigma^5}{120} c_0^{\vee} + ...$$

From (2) one has:

$$\begin{aligned} c''' &= kc' \\ c'^{\vee} &= k'c' + kc'' \\ c^{\vee} &= (k'' + k^2)c' + 2k'c'' \end{aligned}$$

and finally

$$c(\sigma) = (\sigma + k_0\frac{\sigma^3}{6} + k_0'\frac{\sigma^4}{24} + ...)c_0' + (\frac{\sigma^2}{2} + k_0\frac{\sigma^4}{24} + k_0'\frac{\sigma^5}{120} + ...)c_0''$$

Fix coordinates (x, y) on $\mathbf{RP}^2$ generated respectively by vectors $c'(0)$ and $c''(0)$ (see fig.3).

Consider the following conic:

$$x^2 - 2y + k_0 y^2$$

Satisfy the coordinates of $c(\sigma)$ to function $F(x, y) = x^2 - 2y + k_0 y^2$. One obtains:

$$F(x(\sigma), y(\sigma)) = k_0' \frac{\sigma^5}{20} + ...$$

Thus, the conic has a contact of order 5 with c and so this is the *osculating conic* to c. If $k_0' = 0$, then the order of contact is 6. The proposition is proven.

4.2. Proof of the six vertices theorem. Let us show how the six vertices theorem follows from Theorem 1.

Lemma 3. *The parameter $h(x)$ in equation (3) satisfies the following condition:*

$$\int_{S^1} \phi_1(x)\phi_2(x)h(x)dx = 0$$

where $\phi_1(x), \phi_2(x)$ are any two solutions of (3).

Proof. Let $\phi(x)$ be a solution of (3), then $\phi h = A_0\phi$. Lemma 3 follows now from the fact that A_0 is antisymmetric. Indeed,

$$\begin{aligned} &\textstyle\int_{S^1} \phi_1(x)\phi_2(x)h(x)dx = \int_{S^1} \phi_1(x)A_0\phi_2(x)dx = \\ &-\textstyle\int_{S^1} A_0(\phi_1(x))\phi_2(x)dx = -\int_{S^1} \phi_1(x)\phi_2(x)h(x)dx = 0 \end{aligned}$$

The six vertices theorem follows now from Theorem 2 and Lemma 1. In fact, the function $h(x)$ is orthogonal to all the products of solutions of a disconjugate equation of order 3. Thus, it has at least 6 distinct zero points on S^1 (Theorem 2). Sextactic points of a locally convex curve $c \subset \mathbf{RP}^2$ coincide with zero points of h (Proposition 1). One obtains, that a closed convex curve on $\mathbf{RP}^2$ has at least 6 distinct sextactic points. The theorem is proven.

4.3. Geometrical properties of sextactic points. Let us give here two geometrical descriptions of sextactic points.

A. Any curve c in general position has almost everywhere a contact of order 5 with its osculating conic. Nondegenerate sextactic points can be characterized by the fact that c does not cross its osculating conic in such points (see fig.4).

B. Dual curves. Let c_1 and c_2 be locally convex curves, take any two points $p_1 \in c_1$ and $p_2 \in c_2$. Then, there exists a projective transformation $Q \in PGL(3, \mathbf{R})$ such that $Qp_2 = p_1$ and the curve Qc_2 has a contact of order ≥ 5 with c_1 in p_1. Let $\bar{c}$ be a *projectively dual curve* to the curve c.We show that c has a contact of order ≥ 5 with $\bar{c}$ in sextactic points.

Lemma 4. *A point p of a locally convex curve $c \subset \mathbf{RP}^2$ is sextactic if and only if c there exists a projective isomorphism $I : \mathbf{RP}^{2*} \stackrel{\cong}{\to} \mathbf{RP}^2$ such that c has a contact of order ≥ 6 with $I(\bar{c})$ in p.*

Proof. Let C be the osculating conic of a locally convex curve c in a point p. Then the dual conic $\bar{C} \in \mathbf{RP}^{2*}$ is the osculating conic of $\bar{c}$. Take an isomorphism $I : \mathbf{RP}^{2*} \to \mathbf{RP}^2$ which maps $\bar{C}$ to C and the point of contact of $\bar{C}$ with $\bar{c}$ to the point of contact of C with c.

After completion of this paper we received the preprint [4] containing the proof of Theorem 1 and its applications to the theory of space curves. We also discovered unpublished results of A. Viro who gave another proof of the six vertices theorem using the Sturm-Tabachnikov approach.

Acknowledgments. It is a pleasure to acknowledge fruitful discussions with V.I. Arnol'd and S. Tabachnikov.

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