Letters in Mathematical Physics **40:** 31–39, 1997.

Extension of the Virasoro and Neveu–Schwarz Algebras and Generalized Sturm–Liouville Operators

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(Received: 2 February 1996)

Abstract. We consider the universal central extension of the Lie algebra Vect$(S^1) \ltimes C^\infty(S^1)$. The coadjoint representation of this Lie algebra has a natural geometric interpretation by matrix analogues of the Sturm–Liouville operators. This approach leads to new Lie superalgebras generalizing the well-known Neveu–Schwarz algebra.

Mathematics Subject Classifications (1991): 17B65, 17B68, 34Lxx.

Key words: Virasoro algebra, Neveu–Schwarz algebra, Sturm–Liouville operators, superalgebras.

1. Introduction

1.1. STURM–LIOUVILLE OPERATORS AND THE ACTION OF Vect(S^1)

Let us recall some well-known definitions (cf., e.g., [9, 8]).

Consider the Sturm–Liouville operator

$$L = -2c\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u(x), \tag{1}$$

where $c \in \mathbb{R}$ and u is a periodic potential $u(x + 2\pi) = u(x) \in C^\infty(\mathbb{R})$.

Let Vect(S^1) be the Lie algebra of a smooth vector field on S^1: $f = f(x)\mathrm{d}/\mathrm{d}x$, where $f(x + 2\pi) = f(x)$, with the commutator

$$\left[f(x)\frac{\mathrm{d}}{\mathrm{d}x}, g(x)\frac{\mathrm{d}}{\mathrm{d}x}\right] = (f(x)g'(x) - f'(x)g(x))\frac{\mathrm{d}}{\mathrm{d}x}.$$

We define a Vect(S^1)-action on the space of Sturm–Liouville operators.

Consider a *1-parameter family* of Vect(S^1) actions on the space of smooth functions $C^\infty(S^1)$:

$$L^{(\lambda)}_{f(x)\mathrm{d}/\mathrm{d}x}\, a(x) = f(x)a'(x) - \lambda f'(x)a(x). \tag{2}$$

NOTATION. (1) *The operator*

$$L^{(\lambda)}_{f(x)\mathrm{d}/\mathrm{d}x} = f(x)\frac{\mathrm{d}}{\mathrm{d}x} - \lambda f'(x)$$

is called the *Lie derivative*.

(2) Denote $\mathcal{F}_\lambda$ as the Vect(S^1)-module structure (2) on $C^\infty(S^1)$.

DEFINITION. The Vect(S^1) action on L is defined by the commutator with the Lie derivative:

$$\left[L_{f(\mathrm{d}/\mathrm{d}x)}, L\right] := L^{(-(3/2))}_{f(\mathrm{d}/\mathrm{d}x)} \circ L - L \circ L^{(1/2)}_{f(\mathrm{d}/\mathrm{d}x)}. \tag{3}$$

The result of this action is a *scalar operator*, i.e. the operator of multiplication by the function

$$\left[L_{f(x)\mathrm{d}/\mathrm{d}}, L\right] = f(x)u'(x) + 2f'(x)u(x) - cf'''(x). \tag{4}$$

Remark. The argument a of the operator (2) has a natural geometric interpretation as a *tensor density* on S^1 of degree $-\lambda$:

$$a = a(x)(\,\mathrm{d}x)^{-\lambda}.$$

One obtains a natural realization of the Sturm–Liouville operator as an operator on tensor densities $L\colon\mathcal{F}_{1/2} \to \mathcal{F}_{-(3/2)}$ (cf. [8]).

1.2. THE COADJOINT REPRESENTATION OF THE VIRASORO ALGEBRA

The *Virasoro algebra* is a unique (up to isomorphism) nontrivial central extension of Vect(S^1). It is given by the Gelfand–Fuchs cocycle

$$c\left(f(x)\frac{\mathrm{d}}{\mathrm{d}x}, g(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) = \int_0^{2\pi} f'(x)g''(x)\,\mathrm{d}x. \tag{5}$$

The Virasoro algebra is therefore a Lie algebra on the space Vect$(S^1) \oplus \mathbb{R}$ with the commutator

$$[(f,\alpha),(g,\beta)] = ([f,g]_{\mathrm{Vect}(S^1)}, c(f,g)).$$

A deep remark of A. A. Kirillov and G. Segal (see [4, 7]) is that *the* Vect(S^1) *action* (4) *coincides with the coadjoint action of the Virasoro algebra*.

Let us give the precise definitions.

Consider the space $C^\infty(S^1) \oplus \mathbb{R}$ and a pairing between this space and the Virasoro algebra

$$\left\langle (u(x),c), \left(f(x)\frac{\mathrm{d}}{\mathrm{d}x},\alpha\right)\right\rangle = \int_0^{2\pi} u(x)f(x)\,\mathrm{d}x + c\alpha.$$

Space $C^\infty(S^1)\oplus\mathbb{R}$ is identified with a part of the dual space to the Virasoro algebra. It is called the *regular part* (see [4]).

DEFINITION. The coadjoint action of the Virasoro algebra on $C^\infty(S^1)\oplus\mathbb{R}$ is defined by

$$\left\langle \mathrm{ad}^*_{(f(\mathrm{d}/\mathrm{d}x),\alpha)}(u(x),\ c), \left(g\frac{\mathrm{d}}{\mathrm{d}x},\beta\right)\right\rangle := -\left\langle (u(x),\ c), \left[\left(f\frac{\mathrm{d}}{\mathrm{d}x},\alpha\right),\left(g\frac{\mathrm{d}}{\mathrm{d}x},\beta\right)\right]\right\rangle .$$

It is easy to calculate the explicit formula. The result is

$$\mathrm{ad}^*_{(f(x)(\mathrm{d}/\mathrm{d}x),\ \alpha)}(u(x),\ c) = \left(L^{(-2)}_{f(x)(\mathrm{d}/\mathrm{d}x)}\ u(x) - cf'''(x),\ 0\right),$$

where $L_f^{(2)}$ is the operator of Lie derivative (2). This action coincides with the Vect(S^1) action (4) on the space of Sturm–Liouville operators.

Remarks. (1) Note that the coadjoint action of the Virasoro algebra is in fact a Vect(S^1)-action (the center acts trivially).

(2) The regular part of the dual space to the Virasoro algebra can be interpreted as a deformation of the Vect(S^1)-module $\mathcal{F}_{-2}$.

2. Central Extension of Vect(S^1) $\ltimes C^\infty(S^1)$

Consider the semi-direct product $\mathcal{G} = \mathrm{Vect}(S^1)\ltimes C^\infty(S^1)$. This Lie algebra has a three-dimensional central extension given by the nontrivial 2-cocycles

$$\begin{aligned}
\sigma_1\left(\left(f\frac{\mathrm{d}}{\mathrm{d}x},\ a\right),\left(g\frac{\mathrm{d}}{\mathrm{d}x},\ b\right)\right) &= \int_{S^1} f'(x)g''(x)\,\mathrm{d}x,\\
\sigma_2\left(\left(f\frac{\mathrm{d}}{\mathrm{d}x},\ a\right),\left(g\frac{\mathrm{d}}{\mathrm{d}x},\ b\right)\right) &= \int_{S^1} (f''(x)b(x) - g''(x)a(x))\,\mathrm{d}x,\\
\sigma_3\left(\left(f\frac{\mathrm{d}}{\mathrm{d}x},\ a\right),\left(g\frac{\mathrm{d}}{\mathrm{d}x},\ b\right)\right) &= 2\int_{S^1} a(x)b'(x)\,\mathrm{d}x.
\end{aligned} \tag{6}$$

Let us denote $\mathfrak{g}$ as the Lie algebra defined by this extension.

As a vector space, $\mathfrak{g} = \mathrm{Vect}(S^1)\ltimes C^\infty(S^1)\oplus\mathbb{R}^3$. The commutator in $\mathfrak{g}$ is

$$\left[\left(f\frac{\mathrm{d}}{\mathrm{d}x},a,\alpha\right),\left(g\frac{\mathrm{d}}{\mathrm{d}x},b,\beta\right)\right] = \left((fg'-f'g)\frac{\mathrm{d}}{\mathrm{d}x},\ fb'-ga',\ \sigma\right), \tag{7}$$

where

$$\alpha = (\alpha_1,\alpha_2,\alpha_3),\qquad \beta = (\beta_1,\beta_2,\beta_3)\in\mathbb{R}^3\quad\text{and}\quad \sigma = (\sigma_1,\sigma_2,\sigma_3)$$

are the 2-cocycles given by formulas (6).

The Lie algebra $\mathfrak{g}$ is well known in physical literature (see [1, 2]). It was shown in [6] that the cocycles (6) define the *universal* central extension* the Lie algebra $\mathrm{Vect}(S^1) \times C^\infty(S^1)$. This means $H^2(\mathrm{Vect}(S^1) \ltimes C^\infty(S^1)) = \mathbb{R}^3$.

In this Letter we define a space of matrix linear differential operators generalizing the Sturm–Liouville operators. This space gives a natural geometric realization of the coadjoint representation of the Lie algebra $\mathfrak{g}$. We hope that such a realization can be useful for the theory of KdV-type integrable systems related to the Lie algebra $\mathfrak{g}$ as well as for studying the coadjoint orbits of $\mathfrak{g}$ (cf. [4] for the Virasoro case). Remark here that some interesting results concerning coadjoint orbits of $\mathfrak{g}$ have been obtained recently in [3].

3. Matrix Sturm–Liouville Operators

DEFINITION. Consider the following matrix linear differential operators on $C^\infty(S^1) \oplus C^\infty(S^1)$:

$$\mathcal{L} = \begin{pmatrix} -2c_1 \dfrac{\mathrm{d}^2}{\mathrm{d}x^2} + u(x) & 2c_2 \dfrac{\mathrm{d}}{\mathrm{d}x} + v(x) \\ -2c_2 \dfrac{\mathrm{d}}{\mathrm{d}x} + v(x) & 4c_3 \end{pmatrix}, \tag{8}$$

where $c_1, c_2, c_3 \in \mathbb{R}$ and $u = u(x), v = v(x)$ are 2π-periodic functions.

The $\mathrm{Vect}(S^1)$ action on the space of operators (8) is defined, as in the case of Sturm–Liouville operators (1), by commutation with the Lie derivative. We consider $\mathcal{L}$ as an operator on $\mathrm{Vect}(S^1)$ modules:

$$\mathcal{L} : \mathcal{F}_{1/2} \oplus \mathcal{F}_{-(1/2)} \to \mathcal{F}_{-(3/2)} \oplus \mathcal{F}_{-(1/2)}.$$

We will show that there exists a structure on the space of operators (8). Namely, we will define an action of the semi-direct product $\mathrm{Vect}(S^1) \ltimes C^\infty(S^1)$.

3.1. $\mathrm{Vect}(S^1) \ltimes C^\infty(S^1)$-MODULE STRUCTURE

Let us define a 1-parameter family of $\mathrm{Vect}(S^1) \ltimes C^\infty(S^1)$-modules on the space $C^\infty(S^1) \oplus C^\infty(S^1)$:

$$T^{(\lambda)}_{(f(x)\mathrm{d}/\mathrm{d}x,\, a(x))} \begin{pmatrix} \phi(x) \\ \psi(x) \end{pmatrix} = \begin{pmatrix} L^{(\lambda)}_{f(\mathrm{d}/\mathrm{d}x)}\, \phi(x) \\ L^{(\lambda-1)}_{f(\mathrm{d}/\mathrm{d}x)}\, \psi(x) - \lambda a'(x)\phi(x) \end{pmatrix}, \tag{9}$$

where $\phi(x), \psi(x) \in C^\infty(S^1)$. Verify that this formula defines a $\mathrm{Vect}(S^1) \ltimes C^\infty(S^1)$-action:

$$\left[T^{(\lambda)}_{(f(\mathrm{d}/\mathrm{d}x),\, a)}, T^{(\lambda)}_{(g(\mathrm{d}/\mathrm{d}x),\, b)}\right] = T^{(\lambda)}_{((fg'-f'g)\mathrm{d}/\mathrm{d}x,\, fb'-ga')}.$$

* It makes sense, since $H_1(\mathrm{Vect}(S^1) \ltimes C^\infty(S^1)) = 0$.

DEFINITION. Define the Vect$(S^1) \ltimes C^\infty(S^1)$ action on the space of the operators (8) by

$$\left[T_{(f(\mathrm{d}/\mathrm{d}x),a)}, \mathcal{L}\right] := T^{(-1/2)}_{(f(\mathrm{d}/\mathrm{d}x),a)} \circ \mathcal{L} - \mathcal{L} \circ T^{(1/2)}_{(f(\mathrm{d}/\mathrm{d}x),a)}. \tag{10}$$

Let us give the explicit formula of this action.

PROPOSITION 1. *The result of the action* (10) *is an operator of multiplication by the matrix*

$$\left[T_{(f(\mathrm{d}/\mathrm{d}x),a)}, \mathcal{L}\right] = \begin{pmatrix} fu' + 2f'u - c_1 f''' & fv' + f'v - c_2 f'' \\ \quad +va' + c_2 a'' & \quad +2c_3 a' \\ fv' + f'v - c_2 f'' & 0 \\ \quad +2c_3 a' & \end{pmatrix}. \tag{11}$$

Proof. Straightforward.

The following result clarifies the nature of definition (10). It turns out that, in the case of the Lie algebra $\mathfrak{g}$, the situation is analogous to those in the Virasoro case: one obtains a generalization of the Kirillov–Segal result.

THEOREM 1. *The action* (10) *coincides with the coadjoint action of the Lie algebra* $\mathfrak{g}$.

We will prove this theorem in the next section.

3.2. COADJOINT REPRESENTATION OF THE LIE ALGEBRA $\mathfrak{g}$

Let us calculate the coadjoint action of the Lie algebra $\mathfrak{g}$.

DEFINITION. Define the *regular part* of the dual space $\mathfrak{g}^*$ to the Lie algebra $\mathfrak{g}$ as follows (cf. [4]). Put $\mathfrak{g}^*_{\text{reg}} = C^\infty(S^1) \oplus C^\infty(S^1) \oplus \mathbb{R}^3$ and fix the pairing $\langle\ ,\ \rangle$: $\mathfrak{g}^*_{\text{reg}} \otimes \mathfrak{g} \to \mathbb{R}$:

$$\left\langle (u(x),\ v(x),\ \mathbf{c}), \left(f(x)\frac{\mathrm{d}}{\mathrm{d}x},\ a(x),\ \alpha\right)\right\rangle$$

$$= \int_{S^1} f(x)u(x)\,\mathrm{d}x + \int_{S^1} a(x)v(x)\,\mathrm{d}x + \alpha\cdot\mathbf{c},$$

where $\mathbf{c} = (c_1, c_2, c_3),\ \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$.

PROPOSITION 2. *The coadjoint action of* $\mathfrak{g}$ *on the regular part of its dual space* $\mathfrak{g}^*_{\mathrm{reg}}$ *is given by*

$$\mathrm{ad}^*_{(f(\mathrm{d}/\mathrm{d}x),a)}\begin{pmatrix} u \\ v \\ \mathbf{c} \end{pmatrix} - \begin{pmatrix} fu' + 2f'u - c_1 f''' + va' + c_2 a'' \\ fv' + f'v - c_2 f'' + 2c_3 a' \\ 0 \end{pmatrix} \tag{12}$$

where $\mathbf{c} = (c_1, c_2, c_3)$ *(the center of* $\mathfrak{g}$ *acts trivially).*

Proof. By definition of the coadjoint action,

$$\left\langle \mathrm{ad}^*_{(f(\mathrm{d}/\mathrm{d}x),a)}(u, v, \mathbf{c}), \left(g\frac{\mathrm{d}}{\mathrm{d}x}, b\right)\right\rangle = -\left\langle (u, v, \mathbf{c}), \left[\left(f\frac{\mathrm{d}}{\mathrm{d}x}, a\right), \left(g\frac{\mathrm{d}}{\mathrm{d}x}, b\right)\right]\right\rangle .$$

Integrate by part to obtain the result.

The right-hand side of formula (12) coincides with the action (10) of the Lie algebra $\mathrm{Vect}(S^1) \ltimes C^\infty(S^1)$ on space of operators (8).

Theorem 1 follows now from Proposition 1.

Remark. As a $\mathrm{Vect}(S^1)$ module, $\mathfrak{g}^*_{\mathrm{reg}}$ is a deformation of the module $\mathcal{F}_{-2} \oplus \mathcal{F}_{-1} \oplus \mathbb{R}^3$ (and coincides with it if $c_1 = c_2 = 0$). Therefore, the dual space to the Lie algebra has the following tensor sense:

$$u = u(x)(\,\mathrm{d}x)^2, \qquad v = v(x)\,\mathrm{d}x.$$

The space of matrix Sturm–Liouville operators (8) gives a natural geometric realization of the dual space to the Lie algebra $\mathfrak{g}$.

4. Generalized Neveu–Schwarz Superalgebra

We introduce here a Lie superalgebra which contains $\mathfrak{g}$ as its even part. The relation between $\mathfrak{g}$ and this superalgebra is the same as between the Virasoro algebra and the Neveu–Schwarz superalgebra. We show that the differential operator (8) appears as a part of the coadjoint action of the constructed Lie superalgebra.

We follow here the Kirillov method (see [5]) where the Sturm–Liouville operator is realized as the even part of the coadjoint action of the Neveu–Schwarz superalgebra.

4.1. DEFINITION

Consider the $\mathbf{Z}_2$-graded vector space $\mathcal{S} = \mathcal{S}_0 \oplus \mathcal{S}_1$, where $\mathcal{S}_0 = \mathfrak{g} = \mathrm{Vect}(S^1) \oplus C^\infty(S^1) \oplus \mathbb{R}^3$ and $\mathcal{S}_1 = C^\infty(S^1) \oplus C^\infty(S^1)$. Define the structure of a Lie superalgebra on $\mathcal{S}$.

(1) Define the action of the even part $\mathcal{S}_0$ on $\mathcal{S}_1$ by

$$\left[\left(f(x)\frac{\mathrm{d}}{\mathrm{d}x},\ a(x)\right),\ (\phi(x),\ \alpha(x))\right] := T^{(1/2)}_{(f(x)\mathrm{d}/\mathrm{d}x,\ a(x))}(\phi(x),\ \alpha(x))$$

so that, as a Vect(S^1)-module, $\mathcal{S}_1 = \mathcal{F}_{1/2} \oplus \mathcal{F}_{-(1/2)}$.

(2) The even part $\mathcal{S}_0$ acts on $\mathcal{S}_1$ according to (9). Let us define the *anticommutator* $[\ ,\ ]_+ : \mathcal{S}_1 \otimes \mathcal{S}_1 \to \mathcal{S}_0$

$$\Big[(\phi,\ \alpha),\ (\psi,\ \beta)\Big]_+ = \left(\phi\psi\frac{\mathrm{d}}{\mathrm{d}x},\ \phi\beta + \alpha\psi,\ \sigma_+\right), \tag{13}$$

where $\sigma_+ = (\sigma_{+1}, \sigma_{+2}, \sigma_{+3})$ is the continuation of the cocycles (6) to the even part of $\mathcal{S}_0 \subset \mathcal{S}$ defined by the formulæ:

$$\begin{aligned}
\sigma_{+1}((\phi,\alpha),(\psi,\beta)) &= 2\int_{S^1} \phi'(x)\psi'(x)\,\mathrm{d}x,\\
\sigma_{+2}((\phi,\alpha),(\psi,\beta)) &= -2\int_{S^1} (\phi'(x)\beta(x) + \alpha(x)\psi'(x))\,\mathrm{d}x,\\
\sigma_{+3}((\phi,\alpha),(\psi,\beta)) &= 4\int_{S^1} \alpha(x)\beta(x)\,\mathrm{d}x.
\end{aligned} \tag{14}$$

THEOREM 2. *$\mathcal{S}$ is a Lie superalgebra.*

Proof. One must verify the Jacobi identity

$$(-1)^{|X||Z|}[X,[Y,Z]] + (-1)^{|X||Y|}[Y,[Z,X]] + (-1)^{|Y||Z|}[Z,[X,Y] = 0, \tag{15}$$

where $|X|$ is a degree of X ($|X| = 0$ for $X \in \mathcal{S}_0$ and $|X| = 1$ for $X \in \mathcal{S}_1$).

Let us prove (15) for $X, Y, Z \in \mathcal{S}_1$. Take $X = (\phi,\alpha),\ Y = (\psi,\beta),\ Z = (\tau,\gamma)$, then

$$\text{(a)}\quad [(\phi,\alpha),[(\psi,\beta),(\tau,\gamma)]] = -T^{1/2}_{[(\psi,\beta),(\tau,\gamma)]_+}(\phi,\alpha).$$

Since the expression $[(\psi,\beta),(\tau,\gamma)]_+$ is given by (15), one gets $T^{1/2}_{[(\psi,\beta),(\tau,\gamma)]_+}(\phi,\alpha) = T^{1/2}_{(\psi\tau,\psi\gamma+\beta\tau)}(\phi,\alpha)$. According to (9),

$$T^{1/2}_{(\psi\tau,\psi\gamma+\beta\tau)}(\phi,\alpha) = (L^{1/2}_{\psi\tau}(\phi),\ L^{-1/2}_{\psi\tau}(\alpha) - \tfrac{1}{2}(\psi\gamma+\beta\tau)'\phi),$$

where

$$L^{1/2}_{\psi\tau}(\phi) = \psi\tau\phi' - \tfrac{1}{2}(\psi'\tau + \psi\tau')\phi$$

and

$$L^{-1/2}_{\psi\tau}(\alpha) - \tfrac{1}{2}(\psi\gamma+\beta\tau)'\phi = \psi\tau\alpha' + \tfrac{1}{2}(\psi\tau)'\alpha - \tfrac{1}{2}(\psi\gamma)'\phi - \tfrac{1}{2}(\beta\tau)'\phi.$$

In the same way, we obtain

$$\text{(b)}\quad [(\psi,\beta),[(\tau,\gamma),(\phi,\alpha)]] = (L^{1/2}_{\phi\tau}(\psi),\ L^{-1/2}_{\phi\tau}(\beta) - \tfrac{1}{2}(\tau\alpha + \phi\gamma)'\psi),$$

where

$$L^{1/2}_{\phi\tau}(\psi) = \phi\tau\psi' - \tfrac{1}{2}(\phi'\tau + \phi\tau')\psi$$

and

$$L^{-1/2}_{\phi\tau}(\beta) - \tfrac{1}{2}(\tau\alpha + \phi\gamma)'\psi = \phi\tau\beta' + \tfrac{1}{2}(\phi\tau)'\beta - \tfrac{1}{2}(\tau\alpha)'\psi - \tfrac{1}{2}(\gamma\phi)'\psi.$$

For the last term, one has

$$\text{(c)}\quad [(\tau,\gamma),[(\phi,\alpha),(\psi,\beta)]] = (L^{1/2}_{\phi\psi}(\tau),\ L^{-1/2}_{\phi\psi}(\gamma) - \tfrac{1}{2}(\phi\beta + \psi\alpha)'\tau),$$

where

$$L^{1/2}_{\phi\psi}(\tau) = \phi\psi\tau' - \tfrac{1}{2}(\phi'\psi + \phi\psi')\tau$$

and

$$L^{-(1/2)}_{\phi\psi}(\gamma) - \tfrac{1}{2}(\phi\beta + \psi\alpha)'\tau = \phi\psi\gamma' + \tfrac{1}{2}(\phi\psi)'\gamma - \tfrac{1}{2}(\phi\beta)'\psi - \tfrac{1}{2}(\alpha\psi)'\tau.$$

Taking the sum (a) + (b) + (c), one obtains zero.
The proof of the Jacobi identity for the other cases is analogous.

Theorem 2 is proven.

PROPOSITION 3. *The coadjoint action of $\mathcal{S}$ is given by the formula*

$$\mathrm{ad}^*\begin{pmatrix} f\frac{\mathrm{d}}{\mathrm{d}x} \\ a \\ \phi(\,\mathrm{d}x)^{-\frac{1}{2}} \\ \alpha(\,\mathrm{d}x)^{\frac{1}{2}} \end{pmatrix}\begin{pmatrix} u \\ v \\ \mathbf{c} \\ \psi \\ \beta \end{pmatrix} = \begin{pmatrix} L^{(-2)}_f(u) + va' + c_2a'' - c_1f''' \\ \quad +\frac{1}{2}\psi'\phi + \frac{3}{2}\psi\phi' - \frac{1}{2}\beta'\alpha + \frac{1}{2}\beta\alpha' \\ L^{(-1)}_f(v) + 2c_3a' - c_2f'' \\ \quad +\frac{1}{2}\beta'\phi + \frac{1}{2}\beta\phi' \\ 0 \\ L^{(-3/2)}_f(\psi) + \frac{1}{2}a'\beta \\ \quad -2c_1\phi'' + u\phi + v\alpha + 2c_2\alpha' \\ L^{(-1/2)}_f(\beta) \\ \quad -2c_2\phi' + v\phi + 4c_3\alpha \end{pmatrix},$$

where $\mathbf{c} = (c_1, c_2, c_3)$ *(as usual, the center acts trivially).*

Proof. Direct calculation using the definition of the superalgebra S.

In particular, one obtains the following corollary.

COROLLARY.

$$\mathrm{ad}^*_{\begin{pmatrix} 0 \\ 0 \\ \phi(\,\mathrm{d}x)^{-(1/2)} \\ \alpha(\,\mathrm{d}x)^{1/2} \end{pmatrix}} \begin{pmatrix} u \\ v \\ \mathbf{c} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -2c_1\phi'' + u\phi + v\alpha + 2c_2\alpha' \\ -2c_2\phi' + v\phi + 4c_3\alpha \end{pmatrix}$$

This corollary gives the matrix operator (8) defined in Section 2.

The Lie superalgebra $\mathcal{S}$ seems to be an interesting generalization of the Neveu–Shwartz superalgebra. It would be interesting to obtain some information about its representations, coadjoint orbits, corresponding integrable systems, etc.

References

1. Arbarello, E., De Concini, C., Kac, V. G. and Procesi, C.: Moduli space of curves and representation theory, *Comm. Math. Phys.* **117** (1988), 1–36.
2. Harnad, J. and Kupershmidt, B. A.: Symplectic geometries on T^*G, Hamiltonian group actions and integrable systems, *J. Geom. Phys.* **16** (1995), 168–206.
3. Kapoudjian, C.: Etude des orbites coadjointes des extensions de l'algèbre de Virasoro, Mémoire de DEA, Université Lyon-I.
4. Kirillov, A. A.: Infinite-dimensional Lie groups: their orbits, invariants and representations, in: *The Geometry of Moments*, Lecture Notes in Math. 970, Springer-Verlag, New York, 1982, pp. 101–123.
5. Kirillov, A. A.: Orbits of the group of diffeomorphisms of a circle and local superalgebras, *Funct. Anal. Appl.* **15**(2) (1980), 135–137.
6. Ovsienko, V. Yu and Roger, C.: Extension of Virasoro group and Virasoro algebra by modules of tensor densities on S^1, to appear in *Funct. Anal. Appl.*
7. Segal, G. B.: Unitary representations of some infinite-dimensional groups, *Comm. Math. Phys.* **80**(3) (1981), 301–342.
8. Turin, A. N.: On periods of quadratic differentials, *Russian Math. Surveys* **33**(6) (1978), 169–221.
9. Wilczynski, E. J.: *Projective Differential Geometry of Curves and Ruled Surfaces*, Teubner, Leipzig, 1906.