

# Hopf Fibrations and Hurwitz-Radon Numbers

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This column is a place for those bits of contagious mathematics that travel from person to person in the community, because they are so elegant, surprising, or appealing that one has an urge to pass them on. Contributions are most welcome. opf fibration is a beautiful topological construction representing a 3-dimensional sphere as a disjoint union of pairwise linked great circles. Judging by its presence on YouTube, it is almost an object of mass culture; see [23] for a sampler. For visualization, we also strongly recommend the film "Dimensions" [3].

Hopf fibration was defined and studied by the famous German geometer and topologist Heintz Hopf in 1931 [16].

In a sense, contemporary algebraic topology has grown up with the Hopf fibration: the development of the theory of characteristic classes, homotopy theory, and *K*-theory was much influenced by the study of Hopf fibration; see [4, 5, 8, 13, 14]. Hopf fibration appears in other areas of mathematics and physics, including fluid dynamics, gauge theories, cosmology, and elementary particles [22].

In fact, there exist four fiber bundles, called the *Hopf fibrations*, whose fibers, total spaces, and bases are spheres:



The vertical arrows denote projections whose fibers, that is, inverse images of points, are represented by the horizontal arrows. The term "fibration" means that, locally, the total space is the product of the base and the fiber, hence the bigger spheres are filed with smaller ones. For instance, through every point of  $S^3$  there passes one circle,  $S^1$ , and different circles do not intersect.

The construction of Hopf fibrations is very simple (we recall it in the following) and is based on the algebras



Heins Hopf

**Figure I.** Heinz Hopf, 1894–1971 (credit: Springer-Verlag, Heidelberg).

Submissions should be uploaded to http://tmin.edmgr.com or sent directly to Sergei Tabachnikov, tabachni@math.psu.edu  $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ ; according to a famous theorem of Adams [1], there are no such fibrations in other dimensions. Note that the "magic" numbers 1, 2, 4, 8 appear twice: as the dimensions of the bases, and also, this time shifted by 1, as the dimensions of the fibers.

This note, based on the work [17], concerns a similar problem: for which pairs (p, n), is  $\mathbb{R}^n$  foliated by *pairwise skew* affine subspaces  $\mathbb{R}^p$ ? Two disjoint affine subspaces are pairwise skew if they do not contain parallel lines.<sup>1</sup> In other words, we consider fiber bundles

with pairwise skew affine fibers. We call them *affine Hopf fibrations*. We shall give an answer in terms of the Hurwitz-Radon function that appeared in topology in the theorem of Adams [2] about vector fields on spheres.

The reason for considering such "skew" foliations is twofold. On the one hand, the central projection of a Hopf bundle from a sphere to Euclidean space yields such a foliation. And, on the other hand, without the skew restriction, the problem is trivial: for any p < n, one can foliate  $\mathbb{R}^n$  by parallel copies of  $\mathbb{R}^p$ .

## **The Hurwitz-Radon Function**

The sequence

is known (A003485, in the Sloane encyclopedia of integral sequences [21]) as the *Hurwitz-Radon function evaluated at powers of* 2. Namely, the Hurwitz-Radon function  $\rho$  :  $\mathbb{N} \to \mathbb{N}$  is defined as follows. Every natural number *N* can be written as  $N = 2^n(2m + 1)$ , and the function  $\rho$  depends only on the dyadic part of *N*, that is,

$$\rho(N) = \rho(2^n),$$

and

$$\rho(N) = \begin{cases} 2n+1, & n \equiv 0 \mod 4\\ 2n, & n \equiv 1, 2 \mod 4\\ 2n+2, & n \equiv 3 \mod 4. \end{cases}$$

The numbers N = 1, 2, 4, 8 are the only numbers for which  $\rho(N) = N$ . The aforementioned formula can be equivalently written as

$$\rho(2^{n+4}) = \rho(2^n) + 8,$$

which is easier to remember.

#### The History of the Hurwitz-Radon Function

The Hurwitz-Radon function was discovered around 1920, independently, by Adolf Hurwitz and Jean Radon  $[12, 18]^2$ . Both were working on the problem of "square identities,"



**Figure 2.** Adolf Hurwitz, 1859–1919, and Johann Radon, 1887–1956 (both photographs: public domain).

or composition of quadratic forms, that is, formulas of the type

$$(a_1^2 + \dots + a_r^2)(b_1^2 + \dots + b_s^2) = c_1^2 + \dots + c_N^2$$

where  $c_1, ..., c_N$  are bilinear forms in  $a_1, ..., a_r$  and  $b_1, ..., b_s$  with real coefficients. The above identity is said to be of size [r, s, N].

For example, a formula of size [4, 4, 4] is Euler's foursquare identity

$$(a_1^2 + a_2^2 + a_3^2 + a_4^2)(b_1^2 + b_2^2 + b_3^2 + b_4^2)$$
  
=  $(a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4)^2$   
+  $(a_1b_2 - a_2b_1 + a_3b_4 - a_4b_3)^2$   
+  $(a_1b_3 - a_2b_4 - a_3b_1 + a_4b_2)^2$   
+  $(a_1b_4 + a_2b_3 - a_3b_2 - a_4b_1)^2.$ 

that corresponds to multiplication of quaternions. Euler's identity is nothing other than the property |a||b| = |ab| in  $\mathbb{H}$ . It was discovered by Leonhard Euler in 1748, almost a hundred years before the discovery of quaternions. Note also that a similar formula of size [2, 2, 2], that corresponds to multiplication of complex numbers, was known to Diophantus, and also appeared in the early VII century in a book of an Indian mathematician Brahmagupta; Fibonacci also used it in his "Book of Squares." There exists also a square identity of size [8, 8, 8], found by a Dutch mathematician Ferdinand Degen in 1818, that corresponds to multiplication of octonions.

In 1898, Hurwitz proved his famous 1, 2, 4, 8 theorem, stating that a square identity of size [N, N, N] can exist only for  $n \in \{1, 2, 4, 8\}$ . This is equivalent to the statement that  $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$  are the only real normed division algebras with unit. He also formulated a problem to characterize triplets  $r, s, N \in \mathbb{N}$  for which there exists a square identity of size [r, s, M]. This problem remains wide open. Traditionally considered as a problem of number theory, it plays an important role in many other areas of mathematics; for more details see [20].

<sup>&</sup>lt;sup>1</sup>Note that if *n*-dimensional space admits pairwise skew *p*-dimensional subspaces, then  $n \ge 2p + 1$ .

<sup>&</sup>lt;sup>2</sup>The paper of Hurwitz was published posthumously.

Hurwitz and Radon proved that a formula of size [r, N, N] exists if and only if  $r \le \rho(N)$ , and this is still the only case in which the Hurwitz problem is solved.

The Hurwitz-Radon function later appeared, sometimes unexpectedly, in many different areas, such as algebra, representation theory, geometry, topology, and combinatorics. The function gives the dimension of irreducible representations of Clifford algebras that are important in mathematics and physics. The Hurwitz-Radon function appeared in the context of multiantennas' wireless communication.

The "exceptional" numbers 1, 2, 4, 8 (or 0, 1, 3, 7) often appear in mathematics and physics. The best-known subjects are: the celebrated results of topology about parallelizable spheres and Hopf fibrations, and the famous theorem of algebra about classification of real normed division algebras. In these subjects, the above numbers appear as the critical dimensions.

#### The Adams Theorem about Vector Fields on Spheres

The theorem of Adams [2] is perhaps the best known, and it is one of the most beautiful applications of the Hurwitz-Radon function.

**THEOREM 1.** The maximal number of vector fields on the sphere  $S^{N-1}$ , linearly independent at each point, is  $\rho(N) - 1$ .

The existence part of this theorem follows from the Hurwitz-Radon construction of square identities. Suppose that we have a square identity of size [r, N, N]. Consider the *N*-vector  $\bar{c} = (c_1, \ldots, c_N)^T$ . Since every  $c_i$  is a bilinear form in *a*'s and *b*'s, we have:

$$\bar{c} = (a_1 A_1 + \dots + a_r A_r) \begin{pmatrix} b_1 \\ \vdots \\ b_N \end{pmatrix}, \quad (3)$$

where  $A_1, \ldots, A_r$  are  $N \times N$  matrices with real coefficients. The existence of a square identity implies that, for every nonzero  $\bar{b} \in \mathbb{R}^N$ , the vectors  $A_1\bar{b}, \ldots, A_r\bar{b}$  are linearly independent, that is, that  $\bar{c} \neq 0$  in (3) whenever  $\bar{a} \in \mathbb{R}^r$  is nonzero. Indeed, the right-hand side of the square identity is the square of the norm of  $\bar{c}$ , whereas the left-hand side is the product of the squares of the norms of  $\bar{a}$  and  $\bar{b}$ .

Therefore, the existence of a square identity of size [r, N, N] guarantees the existence of r (linear) vector fields in  $\mathbb{R}^N - \{0\}$ , linearly independent at each point. To obtain (at least) r - 1 independent vector fields on  $S^{N-1}$ , one now restricts the constructed r fields to the round sphere and projects to its tangent plane.

Let us mention that the existence part of the Adams theorem was known since the 1920s; the converse statement is one of the most difficult results of algebraic topology.



Figure 3. The Hopf link: two fibers of the Hopf fibration.

Adams's theorem generalizes the theorem that the spheres  $S^0$ ,  $S^1$ ,  $S^3$ , and  $S^7$  are the only *parallelizable*<sup>3</sup> spheres. This was proved independently in 1958 by Kervaire [15] and by Bott and Milnor [6].

#### **The Classical Hopf Fibrations**

# The Hopf Fibration of $S^3$

The fibers of the Hopf fibration

$$\begin{array}{c} S^1 \longrightarrow S^3 \\ & \downarrow \\ & & \downarrow \\ & & S^2 \end{array}$$

are obtained by the action of  $S^1$ , the group of complex numbers with unit absolute values, in the complex plane  $\mathbb{C}^2$ . The action is given by the formula

$$e^{i\phi}(z_1, z_2) = (e^{i\phi}z_1, e^{i\phi}z_2).$$

Restricting this action to the unit sphere  $S^3 \subset \mathbb{C}^2$  given by the equation

$$|z_1|^2 + |z_2|^2 = 1,$$

one obtains a fibration of  $S^3$  by circles.

Equivalently, one may consider the projection  $\mathbb{C}^2 - \{0\} \rightarrow \mathbb{CP}^1$  that sends a point of the complex plane to the line through this point and the origin. In coordinates,

$$(z_1, z_2) \mapsto (z_1: z_2).$$

Restricting this projection to  $S^3 \subset \mathbb{C}^2$ , and noticing that  $\mathbb{CP}^1 \simeq S^2$ , the Riemann sphere, one obtains the Hopf fibration.

Let us mention three properties of the Hopf fibration:

- every two fibers of the Hopf fibration are linked nontrivially (see Fig. 3);
- (2) every fiber is a great circle in  $S^3$ ;
- (3) these great circles are equidistant from each other.

<sup>&</sup>lt;sup>3</sup>Parallelizability of a manifold means that it admits as many vector fields linearly independent at each point as its dimension. Examples of parallelizable manifolds are Lie groups.

#### **Higher-Dimensional Hopf Fibrations**

The construction of the Hopf fibration of  $S^3$  has two immediate generalizations.

Let us replace the algebra of complex numbers  $\mathbb{C}$  by the other division algebras,  $\mathbb{R}$ ,  $\mathbb{H}$ , and  $\mathbb{O}$ . The projections from the quaternionic and octonionic planes to the respective projective lines,  $\mathbb{H}^2 - \{0\} \to \mathbb{HP}^1$  and  $\mathbb{O}^2 - \{0\} \to \mathbb{OP}^1$ , are defined as in the complex case. Restricted to the spheres  $S^7 \subset \mathbb{H}^2$  and  $S^{15} \subset \mathbb{O}^2$ , these projections give rise to the following fibrations

$$S^7 \longrightarrow \mathbb{HP}^1$$
,  $S^{15} \longrightarrow \mathbb{OP}^1$ 

whose fibers are  $S^3$  and  $S^7$ , respectively (the quaternions and octonions of absolute value 1). Note that  $\mathbb{HP}^1 \simeq S^4$  and  $\mathbb{OP}^1 \simeq S^8$  since these are quaternionic and octonionic lines completed by one point at infinity.<sup>4</sup> Finally, replacing  $\mathbb{C}$  by  $\mathbb{R}$ , one obtains the fibration  $S^0 \to S^1 \to \mathbb{RP}^1$  whose fibers consist of two points. This provides a complete list of Hopf fibrations (1).

Another straightforward generalization of the Hopf fibration is obtained by replacing the plane by (m + 1)-dimensional complex vector space. Consider the action of  $S^1$  in  $\mathbb{C}^{m+1}$  given by

$$e^{i\phi}(z_1, z_2, \dots, z_{m+1}) = \left(e^{i\phi}z_1, e^{i\phi}z_2, \dots, e^{i\phi}z_{m+1}\right)$$

Restricting this action to the sphere  $S^{2m+1} \subset \mathbb{C}^{m+1}$ , one obtains a fibration of  $S^{2m+1}$  by  $S^1$ . The base of this fibration is the complex projective space  $\mathbb{CP}^m$ .

The same construction can be applied in the quaternionic cases, replacing  $S^1$  by the unit sphere  $S^3 \subset \mathbb{H}$ . One obtains three infinite series of fibrations:



This construction does not work in the octonionic case, and the fibration with fibers  $S^7 \rightarrow S^{15} \rightarrow S^8$  remains exceptional and does not belong to any series.

It is still true that every fiber of a fibration (4) is a great sphere.

#### **Projection to Euclidean Spaces**

The usual way to "visualize" the Hopf fibration is by way of projecting them to the Euclidean spaces.

The *stereographic projection* is ubiquitous. This map identifies the sphere with one point, the North Pole *P*, deleted, with *n*-dimensional Euclidean space (the tangent space at the South pole); see Figure 4. The main property of the stereographic projection is that this is a conformal map: it preserves the angles. It sends circles to circles, except for



**Figure 4.** The stereographic projection and the central projection: conformal geometry vs projective geometry.

the circles that pass through *P*: they are sent to straight lines.

An equally important projection from the *n*-dimensional sphere to the *n*-dimensional Euclidean space is the *central projection*. It projects the sphere  $S^n \subset \mathbb{R}^{n+1}$ , with the equatorial sphere  $S^{n-1}$  removed, from the center *C* of the sphere to an affine space  $\mathbb{R}^n \subset \mathbb{R}^{n+1}$  that does not pass through the origin. The central projection sends great circles to straight lines, and great spheres to affine planes.

The image of the Hopf fibration of  $S^3$  under the stereographic projection is a beautiful geometric structure in  $\mathbb{R}^3$  called the Villarceau circles. These circles foliate  $\mathbb{R}^3$  with a straight line removed (this line is the stereographic image of the Hopf circle through point *P*). See Figures 5 and 6.

The image of the Hopf fibration of  $S^3$  under the central projection is a fibration of  $\mathbb{R}^3$  by lines; see Figure 7. These lines are pairwise skew: two parallel lines "intersect" at infinity, and their preimages in  $S^3$  are great circles that intersect at a point on the equator.

**EXAMPLE 2.1.** The central projections of the fibrations (1) and (4) give fibrations of the Euclidean space by pairwise skew affine subspaces:

$$\mathbb{R}^p \longrightarrow \mathbb{R}^n \longrightarrow \mathbb{R}^{n-p}, \quad \text{where} \quad (p,n) = \begin{cases} (1, 2m+1), \\ (3, 4m+3), \\ (7, 15). \end{cases}$$

#### **Affine Hopf Fibrations**

Let us consider the affine Hopf fibrations (2) in full generality.

## **Existence Theorem and First Examples**

The problem of describing all pairs (p, n) for which such fibrations exist was solved in [17].<sup>5</sup>

**THEOREM 2.**  $\mathbb{R}^n$  admits a fibration by p-dimensional pairwise skew affine subspaces if and only if  $p \le \rho(n-p) - 1$ .

<sup>4</sup>See [7] concerning octonions and  $\mathbb{OP}^1$ .

<sup>&</sup>lt;sup>5</sup>The next step would be a classification of affine Hopf fibrations. See [10, 19] for partial results. A classification of fibrations of *S*<sup>3</sup> by great circles was obtained by H. Gluck and F. Warner [9]: they are in one-to-one correspondence with distance-decreasing self-maps of a 2-sphere.



**Figure 5.** A key-ring model of Villarceau circles (Wikimedia Commons).



**Figure 6.** Villarceau circles and skew lines at Musée de l'Oeuvre Notre-Dame, Strasbourg (photograph by Luba Shenderova-Fock).



**Figure 7.** "Nested hyperboloids" by David Eppstein (from the Wikipedia article "Skew Lines").

The condition of this theorem is satisfied in Examples 2.1. Indeed, if p = 1 and n = 2m + 1, then n - p is even, so that  $\rho(n-p) \ge 2$ . If p = 3 and n = 4m + 3, then n - p is a multiple of 4, so that  $\rho(n-p) \ge 4$ . Finally, if p = 7 and n = 15, then  $\rho(n-p) = 8$ .

Let us sketch the proof of Theorem 2. The existence part is, again, a consequence of the Hurwitz-Radon construction. Given a square identity of size [r, N, N], rewrite the vector of bilinear forms  $\bar{c} = (c_1, \ldots, c_N)^T$  of the right-hand side in the following form

$$\bar{c} = (b_1 B_1 + \dots + b_N B_N) \begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix}$$

where  $B_1, \ldots, B_N$  are  $N \times r$  matrices with real coefficients. Using the notation

$$B(\bar{b}) := b_1 B_1 + \dots + b_N B_N,$$

the square identity implies that  $B(\overline{b})$  is of *maximal rank* (i.e., of rank *r*) for all  $\overline{b} \neq 0$ . This matrix expression is of course equivalent (actually, dual) to (3).

Furthermore, using a linear change of coordinates  $(b_1, \ldots, b_N)$ , we can assume that the last column of each matrix  $B_i$  is the vector  $(0, \ldots, 0, 1, 0, \ldots, 0)^T$  with 1 at *i*th position. Let  $B'_i$  denote the  $N \times (r-1)$  matrix  $B_i$  with the last column removed, and let

$$B(\bar{b})' := b_1 B'_1 + \dots + b_N B'_N.$$

Our construction follows. Consider the direct product

$$\mathbb{R}^N \times \mathbb{R}^{r-1} \simeq \mathbb{R}^{N+r-1}.$$

with  $\mathbb{R}^N$  being understood as the "vertical," and  $\mathbb{R}^{r-1}$  as the "horizontal" subspace; the coordinates on these spaces are denoted by  $y = (y_1, \ldots, y_N)$  and  $x = (x_1, \ldots, x_{r-1})$ , respectively. At every point  $\overline{b} \in \mathbb{R}^N$ , consider the affine (r-1)-dimensional subspace of  $\mathbb{R}^{N+r-1}$  through  $\overline{b}$  defined by

$$y = B(\bar{b})'x + \bar{b}.$$

It follows from the maximality of the rank that any two of these affine spaces are skew, and we obtain an affine Hopf fibration on  $\mathbb{R}^{N+r-1}$ .

Next we give geometric arguments for the nonexistence. Consider the first example after dimension three, that of  $\mathbb{R}^4$ .

**EXAMPLE 3.1.** The space  $\mathbb{R}^4$  has no affine Hopf fibrations. Indeed, the only affine subspaces of  $\mathbb{R}^4$  that can be pairwise skew are 1-dimensional. However, if p = 1, then n - p = 3 is odd, so  $\rho(n - p) = 1$ . The condition of Theorem 2 is not satisfied. The same holds true for all spaces of dimension  $2^k$ , that is, the space  $\mathbb{R}^{2^n}$  admits no affine Hopf fibrations.

For the proof, consider first  $\mathbb{R}^3$  equipped with the fibration depicted in Figure 7. Choose any fiber, say vertical, and a 2-dimensional plane orthogonal to it. Draw a circle around the point of intersection of the plane and the fiber. Pick any point x of the circle and consider the unit vector that belongs to the fiber through x. Project this vector to the tangent line of the circle (the projection goes in two steps: first project to the horizontal plane, then to the tangent); see Figure 8. Since the fiber through x is skew with the vertical line, the projection is nonzero.

Assume now that  $\mathbb{R}^4$  is fibered by pairwise affine lines. Exactly the same construction would then give a



Figure 8. Affine Hopf fibrations and vector fields on spheres.

nonvanishing vector field on a 2-dimensional sphere that lies in a 3-dimensional affine subspace, orthogonal to a fiber. This, however, contradicts the famous *hairy ball theorem*:  $S^2$  does not admit a nonvanishing continuous tangent vector field. It follows that  $\mathbb{R}^4$  admits no affine Hopf fibrations.

A similar argument is used to deduce the "necessary" part of Theorem 2 from Theorem 1.

In fact, Theorem 2 extends to a semilocal result: its assertion holds for fibrations by pairwise skew affine subspaces of a neighborhood of a single fiber.

More generally, a remarkable series of affine Hopf fibrations corresponds to the following dimensions:

$$(p, n-p) = (\rho(2^n) - 1, 2^n).$$

None of them is a projection of a classical Hopf fibration.

#### A Table

The condition of Theorem 2 is implicit (since p appears in both parts of the inequality). Therefore, for a given n, the situation has to be analyzed.

Consider the cases of  $n \le 80$ . The dimensions (p, n) for which affine Hopf fibrations of  $\mathbb{R}^n$  with *p*-dimensional fibers exist are presented in the following tables.

Recall the semilocal extension of Theorem 2. It implies that, given a (local) (p, n) affine Hopf fibration, the intersection with a hyperplane transverse to the fibers yields a (local) (p - 1, n - 1) affine Hopf fibration. This leads to the notion of a *dominant* (p, n) affine Hopf fibration: it is the case when a (p + 1, n + 1) affine Hopf fibration does not exist. In the table that follows, the dominant pairs (p, n) are shown in boldface.

**EXAMPLE 3.2.** Let us now consider p = 8 and n = 24. Since  $\rho(16) = 9$ , the condition of Theorem 2 is satisfied. We therefore have an affine Hopf fibration of  $\mathbb{R}^{24}$  with 8-

The fibrations with n = 3, 7, 15 are the central projections of the Hopf fibrations. The next values are as follows.

n	17	18	19	20	21	22	23	24	25	26	27	28	29	30	<b>31</b>	32
p	1	2	3	4	5	6	7	8	1	2	3	4	5	6	7	
			1		1	2	3				1		1	2	3	
							1								1	

dimensional fibers. This is the first example of an affine Hopf fibration that does not come from a projection of a classical Hopf fibration. including two interesting cases: (p, n) = (8, 24) and (7, 31). The next values are:

n	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
p	1	2	3	4	5	6	7	8	9	2	3	4	5	6	7	
			1		1	2	3		1		1		1	2	3	
							1								1	

We see another example of an interesting "non-Hopf" fibration:  $\mathbb{R}^9 \to \mathbb{R}^{41} \to \mathbb{R}^{32}$ . The next values are as follows.

n	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
p	1	2	3	4	5	6	7	8	1	2	3	4	5	6	7	
			1		1	2	3				1		1	2	3	
							1								1	

and

n	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80
p	1	2	3	4	5	6	7	8	9	10	11	4	5	6	7	
			1		1	2	3		1	2	3		1	2	3	
							1				1				1	

We observe several more non-Hopf fibrations with relatively large fibers.

Another observation is that some of the spaces admit no fibrations with skew fibers at all. In addition to the cases when the dimension is a power of 2, this happens for  $\mathbb{R}^{80}$ .

## **The Complex Case**

The definition of affine Hopf fibrations in  $\mathbb{C}^n$  is exactly the same as in the real case. Surprisingly, the problem is much more difficult in this case, and it seems not to be reducible to linear algebra.

Only necessary conditions are known for the existence of complex affine Hopf fibrations. It was proved in [10] that an affine Hopf fibration of  $\mathbb{C}^n$  with fibers of dimension pmay exist only if, for each integer r with  $0 \le r \le p$ , the coefficients of  $t^r$  in the power series expansion of

$$\left(\frac{t}{\ln(1+t)}\right)^{n-1}$$

is an integer.

The minimal dimension n, considered as a function of p, grows very quickly. For example, for a complex skew fibration with 1-dimensional fibers, the ambient space is at least 3-dimensional (like in the real case), but for 2-dimensional fibers, the ambient dimension is not less than 26, and for 4-dimensional fibers, the ambient dimension is not less than 2884.

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