SYMPLECTIC LEAVES OF THE GEL'FAND–DIKII BRACKETS AND HOMOTOPY CLASSES OF NONDEGENERATE CURVES

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1. INTRODUCTION

In [1, 2] Gel'fand and Dikii have defined a pair of Poisson brackets on the space of differential operators of arbitrary order with peridic coefficients. In the present article we study the symplectic leaves of the Gel'fand–Dikii brackets, associated with the Lie groups GL(n), SL(n), Sp(2k), and SO(2k + 1).

It turns out that the monodromy operator is the only continuous invariant of a symplectic leaf of the second bracket and the contiguity of leaves coincides with the contiguity of the conjugacy classes of the matrix groups, numbering the Gel'fand–Dikii brackets, the symplectic leaves of the first bracket form trivial bundles of finite [for SL(n), Sp(2k), and SO(2k + 1)] or infinite [for GL(n)] codimension with contractible fiber (Sec. 2). The study of discrete invariants of leaves of the second bracket can be reduced to the homotopy classification of nondegenerate curves in $\mathbf{P}^{n-1}$. A differential operator of n-th order defines a nondegenerate curve in $\mathbf{P}^{n-1}$ (the projectivization of its "solution cure"), and two differential operators belong to a single symplectic leaf when the corresponding curves are homotopic (see Sec. 3). For differential operators of third order this approach gives a complete classification of leaves (using Little's results [3] on nondegenerate curves on S^2). It turns out that the differential operators of n-th order, giving nonoscillatory equations, form a separate symplectic leaf of the second Gel'fand–Dikki bracket. In the proof of this statement we use Shapiro's theorem [4] on nonoscillatory curves on S^{2k}. The proofs of the main theorems are given in Sec. 4.

In the elementary case of SL(2), the problem of classification of the symplectic leaves of the second Gel'fand–Dikii bracket is equivalent to three other classification problems: description of orbits of the co-adjoint representation of the Virasoro group, of normal forms of Hille equations, and of types of projective structures on the circle. These problems were solved at different times independently by Kuiper [5], Lazutkin and Pankratov [6], Kirillov [7], and Segal [8]. The continuous invariants of symplectic leaves of the Gel'fand–Dikii bracket, associated with SL(3), were investigated by the first author in [9], where the classification problem for symplectic leaves was connected for the first time with the computation of homotopy classes of nondegenerate curves, and, with the help of a theorem of Little, their classification is obtained in the case of the identity monodromy operator. A complete classification of these leaves has been obtained by the second author and Shapiro (see [22]). In [10] the contact-projective structures on the supercircle and the orbits of the coadjoint representation of the Neveu–Schwartz and the Ramon supergroups have been classified. In [11] the continuous invariants and the contiguities of symplectic leaves of the Gel'fand–Dikii brackets for the general case of an arbitrary semisimple Lie group have been investigated by the method of Hamiltonian reduction. The Gel'fand–Dikii brackets in the general case are introduced on the gauge classes of matrix differential operators, and the monodromy operator of a gauge class turns out to be the only continuous invariant of a symplectic leaf.

Recently the Gel'fand–Dikii brackets have found application in the conformal field theory. Zamolodchikov [12] has independently constructed an infinite-dimensional algebra with quadratic relations that is the quantum analogue of the Gel'fand–Dikii Poisson algebra, associated with SL(3). Luk'yanov and Fateev have suggested a method for the quantization of the Gel'fand–Dikii brackets (see [13] and the bibliography given there). The authors hope that the classification of symplectic leaves turns out to be useful for understanding the sense of quantization of these brackets, since the possibility of restriction

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of this procedure to symplectic leaves is a natural condition for quantization. In the case of the Virasoro algebra, the problem of quantization of symplectic leaves was considered by Witten [14].

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2. THE CONTIGUITY OF SYMPLECTIC LEAVES

2.1. Let us define the Gel'fand–Dikii brackets, associated with the Lie groups GL(n), SL(n), Sp(2k), and SO(2k + 1) (see [15]). In the case of GL(n), they are given on the space $\mathfrak{L}$ of all differential operators L of the form $D^n + \sum_{i=0}^{n-1} u_i(x) D^i$, where $D = d/dx$, $u_i \in C^\infty(S^1, \mathbf{k})$ ($\mathbf{k} = \mathbf{R}$ or $\mathbf{C}$). Let us consider the space of pseudodifferential symbols: $X = \sum_{j=1}^{n} a_j(x) D^{-j}$, $a_j \in C^\infty(S^1, \mathbf{k})$.

Let us associate with each X the linear functional $l_X(L) = \int_{S^1} \mathrm{res}(XL)\, dx$, where res (XL) is the function on S^1 that is defined in the following manner. Using the Leibnitz relation $D^{-1}f = fD^{-1} + \sum_{i=1}^{\infty} (-1)^i f^{(i)} D^{-1-i}$, we can express the product XL as a pseudodifferential operator $\sum_{m\in\mathbf{Z}} p_m D^m$. Then, by definition, $\mathrm{res}(XL) = p_{-1}(x)$. It is clear that that each linear functional on the space $\mathfrak{L}$ can be expressed in the form ℓ_X, where X is a pseudodifferential symbol.

Definition. The operators that associate with a linear functional ℓ_X the vector fields

$$L_X^1 = (LX - XL)_+, \quad L_X^2 = L(XL)_+ - (LX)_+ L \tag{1}$$

respectively, on the space of operators (here the index + denotes the differential part) are called the operators of the first and the second Gel'fand–Dikii Hamiltonian structures. These operators define Poisson brackets on $\mathfrak{L}$: $\{l_X, l_Y\}_i (L) = l_Y(L_X^i)$, called the Gel'fand–Dikki brackets.

For the groups SL(n), Sp(2k), and SO(2k + 1) we introduce the following restrictions on $\mathfrak{L}$: $u_{n-1} \equiv 0$, $L^* = L$, and $L^* = -L$, respectively [here $(\sum u_i D^i)^* = \sum (-1)^i D^i u_i)$]. The restrictions on X are determined explicitly from the following condition: The vector field L_X must be tangential to $\mathfrak{L}$ $(D^n + L_X \in \mathfrak{L})$ (see [15]). The operators of the Hamiltonian structures, associated with these groups, are also given by Eqs. (1), except $L_X^1 = (DLX - XLD)_+$ for $SO(2k+1)$.

2.2. THEOREM 1. The symplectic leaves of the first Gel'fand–Dikii bracket form a trivial bundle in $\mathfrak{L}$ of codimension equal, for simple groups, to the dimension of the Cartan subalgebra [n − 1 for the group SL(n) and k for Sp(2k) and SO(2k + 1)], and of infinite codimension for GL(n).

Remark. To each operator L there corresponds the differential equation $L\psi = 0$ with periodic coefficients. Its monodromy operator determines the conjugacy class in the corresponding matrix Lie group (two monodromy operators can be compared only up to conjugacy, since they act in different spaces).

THEOREM 2. The monodromy operator is the only local invariant of a symplectic leaf of the second Gel'fand–Dikii bracket, associated with one of the groups GL(n), SL(n), Sp(2k), and SO(2k + 1), i.e., each infinitesimal deformation $L_\varepsilon = L + \varepsilon\ell + O(\varepsilon^2)$ (in the space $\mathfrak{L}$) of the differential operator L, preserving its monodromy operator (up to conjugacy), can be expressed in the form $\ell = L_X$.

COROLLARY 1. The codimension in $\mathfrak{L}$ of a symplectic leaf of the second bracket is equal to the codimension in the Lie group of the conjugacy class of the corresponding monodromy operator.

2.3. Definitions (cf. [16]). The germ of a smooth mapping F of a finite-dimensional manifold Λ (the basis of deformation) in $\mathfrak{L}$, such that $F(0) = L$ is called a deformation of the operator L. Two deformations $F_0(\lambda)$ and $F_1(\lambda)$ are said to be equivalent if there exists a homotopy F_τ, smoothly depending on g, that connects F_0 and F_1 and is such that the germ (with respect to λ) of the vector field $dF_\tau(\lambda)/d\tau$ is tangential to the symplectic leaves for all

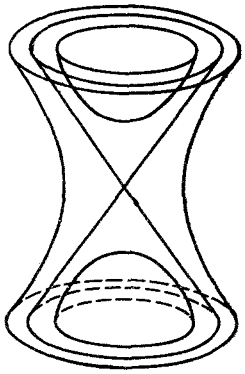

Fig. 1

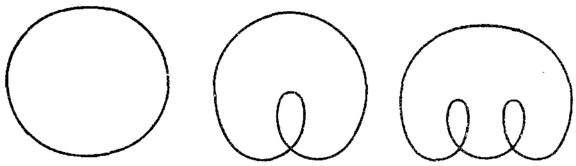

Fig. 2

$\tau \in (0, 1)$. A deformation F_0 is said to be induced from F_1 if F_1 is obtained from F_1 by a change of the parameter. A deformation F of the operator L is said to be versal if each deformation of L is equivalent to some one, induced from F.

THEOREM 3. Each versal deformation of the operator L is finite-dimensional and reduces to a versal deformation of the monodromy operator in the corresponding Lie group [GL(n), SL(n), Sp(2k), and SO(2k + 1)]. In other words, the symplectic leaves are contiguous to each other in the same manner as the conjugacy classes (or the orbits of the coadjoint representation) of matrix Lie groups.

COROLLARY 2. The versal deformations of differential operators are, as deformations of matrices, given by the Jordan–Arnol'd normal forms [16].

Example. The contiguity of orbits of the coadjoint representation of the group SL(2, **R** is given in Fig. 1 (cf. [6]).

Thus, the first and the second Gel'fand–Dikii brackets are the analogues of the constant and the linear Poisson structures on Lie coalgebras, although given, respectively, by a linear relation and a quadratic (with respect to L) relation.

3. DISCRETE INVARIANTS OF THE SYMPLECTIC LEAVES OF THE SECOND BRACKET

3.1. Definition 1. A mapping $f: \mathbf{R} \to \mathbf{P}^m$ ($\mathbf{P}^m = \mathbf{RP}^m$ or $\mathbf{CP}^m$) is called a qusiperiodic nondegenerate curve if a) there exists an operator $PM \in \mathrm{PGL}\,(m+1)$, such that $f(x + 2\pi) = PMf(x)$ and b) the accompanying flag of the curve f – the chain of subspaces $F_1 \subset F_2 \subset \ldots \subset F_m$, where $F_k = \langle f^{(1)}, \ldots, f^{(k)} \rangle$, is full, i.e., $F_m = \mathbf{P}^m$.

2. The deformation of a quasiperiodic nondegenerate curve in the class of quasiperiodic nondegenerate curves with given operator PM is called a homotopy of this curve. Example. A homotopy of a closed quasiperiodic nondegenerate curve is a deformation of it in the class of closed quasiperiodic nondegenerate curves (PM = id).

3. The family of the hyperplanes $F_{m-1}(x) \subset \mathbf{P}^m$ that are osculating with f forms the dual quasiperiodic nondegenerate curve f* in $\mathbf{P}^{m*}$. A quasiperiodic nondegenerate curve f in $\mathbf{P}^m$ is self-dual if after an identification of $\mathbf{P}^{m*}$ and $\mathbf{P}^m$ that preserves the projective structure, the curve f*, dual to f, transforms into a curve that is projectively equivalent to f [i.e., $f^* = Af$, $A \in \mathrm{PGL}\,(m+1)$)].

3.2. Definition. To each differential operator L of order n with periodic coefficients there corresponds an equivalence class of quasiperiodic nondegenerate curves in $\mathbf{P}^{n-1}$, determined in the following manner. We associate with each point x on the line a hyperplane in the solution space V of the equation $L\psi = 0$ (or, in other words, the line Γ_x in the dual space V*), that consists of the solutions which vanish at x. By the same token, the curve γ, the projectivization of the family Γ, is defined in the projectivization $PV^* \cong \mathbf{P}^{n-1}$. In the homogeneous coordinates on $\mathbf{P}^{n-1}$ the curve γ is given by the mapping $x \mapsto (\psi_1(x): \ldots : \psi_n(x))$, where $\Psi(x) = (\psi_1(x), \ldots, \psi_n(x))$ is the fundmental system of solutions of the equation $L\psi = 0$. Thus, the curve γ is a quasiperiodic nondegenerate curve a the projeciton from $\mathbf{R}^n$ ($\mathbf{C}^n$) on $\mathbf{P}^{n-1}$ of a quasiperiodic solution curve ψ with the monodromy operator M. Different methods of identification of PV* with $\mathbf{P}^{n-1}$ give equivalent curves γ.

Proposition 1. In the case of SL(n), to the adjoint operator L* there corresponds a quasiperiodic nondegenerate curve, dual to the quasiperiodic nondegenerate curve of the operator L. In particualr, to a (skew) self-adjoint operator there corresponds a self-dual curve.

The proof of this well-known proposition is given in Sec. 4, since we could not find it in the literature.

3.3. THEOREM 4. The complete set of invariants of symplectic leaves of the second Gel'fand–Dikii brackets, associated with the Lie groups GL(n) and SL(n) [or Sp(2k) and SO(2k + 1)] consists of the monodromy operator and the homotopy class of the corresponding quasiperiodic nondegenerate curves (of self-dual quasiperiodic nondegenerate curves).

Proof. Differential operators with the same monodromy are homotopic in the class of these operators if and only if the solution curves (see definition above), corresponding to them, are homotopic. Now Theorem 4 follows from Theorem 2, since the solution curves are homotopic if and only if their projections in $\mathbf{P}^{n-1}$ are homotopic. Indeed, the projection $\gamma \subset \mathbf{P}^{n-1}$ is uniquely lifted to the solution curve Ψ (up to a numerical factor); the dependence on x of the factor, connecting the homogeneous and the linear coordinates, is determined from the condition $dW(\Psi)/dx = u_{n-1}W(\Psi)$ on the Wronskian. In the cases of Sp and SO the homotopy property is understood in the class of self-dual quasiperidic nondegenerate curves in $\mathbf{P}^{n-1}$ (see Proposition 1).

COROLLARY 3 [6, 7]. The orbits of the coadjoint representation of the Virasoro group are numbered by the Jordan form of the monodromy operator from $SL(2, \mathbf{R})$ and one natural-number parameter.

Proof. For $SL(2, \mathbf{R})$ the operator of the second Hamiltonian structure coincides with the operator ad* for the Virasoro algebra (see [17]). To the given monodromy operator there correspond a countable number of curves on $\mathbf{RP}^1 \cong S^1$, distinguished by the number of complete rotations by the period.

Remark. The monodromy operator and the homotopy class form a complete, but "an overlapping" system of invariants, since the homotopy class of quasiperiodic nondegenerate curves is connected with the given conjugacy class of the operator PM. At first, we fix a homotopy class of quasiperiodic nondegenerate curves. In the cases of SL(2k + 1) this is the only invariant of SL(2k + 1) and SO(2k + 1) this is the only invariant of symplectic leaves, since $PGL(2k+1) \cong SL(2k+1)$. For SL(2k) and Sp(2k), to each homotopy class there correspond two leaves with the operator $\pm M$ ($PGL(2k) \cong SL(2k)/\mathbf{Z}_2$), and for GL(n) there corresponds a one-parameter family of leaves.

Let us now discuss the problem on the number of the symplectic leaves that correspond to a given monodromy operator.

Proposition 2. a) (B. Z. Shapiro and Khesin). In the cases of $GL(3, \mathbf{R})$ and $SL(3, \mathbf{R})$, to the monodromy operators, having the Jordan normal forms

$$\begin{pmatrix} \lambda & & 0 \\ & \lambda & \\ 0 & & \mu \end{pmatrix} \text{and} \begin{pmatrix} \lambda & 1 & 0 \\ & \lambda & 0 \\ 0 & & \lambda \end{pmatrix},$$

there correspond two symplectic leaves and to the remaining monodromy operators there correspond three leaves (see [22]).

b) For GL(n), SL(n), and SO(2k + 1), at least two symplectic leaves correspond to an arbitrary monodromy operator M.

c) At least three symplectic leaves correspond to the identity monodromy operator M = id for $GL(2k+1, \mathbf{R})$ and $SL(2k+1, \mathbf{R})$ and also to the monodromy operator M = –id for $GL(2k, \mathbf{R})$ and $SL(2k, \mathbf{R})$.

d) For $Sp(2k, \mathbf{R})$, an infinite number of symplectic leaves correspond to each monodromy oprator.

Proof. To each quasiperiodic nondegenerate curve in $\mathbf{RP}^{n-1}$ there corresponds a quasiperiodic curve in the adjoint group [$GL(n, \mathbf{R})$ or $SL(n, \mathbf{R})$)], given by the phase flow of the corresponding differential equation $L\psi = 0$. In this connection, the monodromy operator is interpreted as an element of the universal covering of a matrix group. Now statement b) follows from the fact that $\pi_1 \geqslant 3\pi_1(GL(n, \mathbf{R})) = \pi_1(SL(n, \mathbf{R})) = \pi_1(SO(n, \mathbf{R})) = \mathbf{Z}_2$. Statement c) follows from the following statement.

Statement 1. Closed nondegenerate nonoscillatory curves in $\mathbf{RP}^{n-1}$ (intersecting each hyperplane $\mathbf{RP}^{n-2} \subset \mathbf{RP}^{n-1}$ in at most n − 1 points) constitute a separate homotopy class. These curves belong to the zero and a nonzero class of $\pi_1(\mathbf{RP}^{n-1})$, respectively, for n odd and even.

In other words, in the space of differential operators the nonoscillatory operators L (for which an arbitrary solution of the corresponding equation $L\psi = 0$ has at most n − 1 zeros on the period) with the monodromy operator M = id (for n = 2k + 1) and M = −id (for n = 2k) form separate symplectic leaves.

The proof of Statement 1 is analogous to the proof of B. Z. Shapiro's theorem on nonoscillatory curves on S^{2k}, given below.

The proof of a) is based on Little's theorem [3] that each closed nondegenerate curve on $\mathbf{RP}^2$ is homotopic to one of the three curves, given in Fig. 2. There also exists the notion of nonoscillation property for nonclosed quasiperiodic nondegenerate curves (M ≠ id), but the number of their homotopy classes can turn out to be less than three. This fact is connected with the existence of the homotopies that change the initial flag of the curve (for a fixed monodromy operator) and join the oscillatory and nonoscillatory nonclosed quasiperiodic nondegenerate curves ([22]). Finally, d) follows from the relation $\pi_1(Sp(2k)) = \mathbf{Z}$. A finer invariant of the symplectic leaves is the Maslov–Arnol'd index of the curve on the Lagrange Grassmanian, averaged over the period. Such a curve is defined by the evolution of a certain Lagrangian plane on the phase space, defined by a given self-adjoint operator. Apparently, this invariant distinguishes symplectic leaves for a fixed monodromy operator.

Let us observe that there do not exist nonoscillatory differential operators of even order with the identity monodromy (a closed quasiperiodic nondegenerate curve on the sphere interacts an arbitrary hyperplane an even number of times with regard for multiplicity).

THEOREM (Shapiro [4]). The closed quasiperiodic nondegenerate curves in $S^{2k} \subset \mathbf{R}^{2k+1}$, that intersect each subspace $\mathbf{R}^{2k} \subset \mathbf{R}^{2k+1}$ at most 2k times form a separate homotopy class among the closed quasiperiodic nondegenerate curves.

Idea of the Proof. Let us suppose that there exists a homotopy in the class of closed nondegenerate curves that connects a nonoscillatory curve with an oscillatory one. Let t_0 be the value of the homotopy parameter at the moment of loss of the nonoscillation property. The set of nonoscillatory curves is open. Therefore, the curve γ_{t_0} is oscillatory. This means that there exists a hyperplane that intersects γ_{t_0} in m (>2k) points (with regard for multiplicity). The corresponding equation has a solution with n zeros and, therefore, a solution with m simple zeros (Hartman's theorem [18]). Therefore, to this solution there corresponds a hyperplane in $\mathbf{R}^{2k+1}$, that has m simple intersections with γ_{t_0}. The curves, close to γ_{t_0}, also intersect this hyperplane in m simple points. Consequently, t_0 is not the first moment of loss of the nonoscillation property. It follows from the results of [19] that the set of nonoscillatory curves is connected.

Remark. This theorem corroborates the statement [20] that for each $n \geqslant 3$ there exists on S^n exactly two homotopy classes (without regard for orientation) of closed nondegenerate curves. These are at least three on S^{2k}. The lifting of a nondegenerate curve in the group SL(2k + 1) defines a $\mathbf{Z}_2$ -invariant − an element of $\pi_1(SL(2k + 1))$. Moreover, the nonoscillatory curves form a separate homotopy class among the curves whose liftings to the group are contractible.

Conjecture (B. Z. Shapiro). The number of hotomopy classes of closed nondegenerate curves in S^n (without regard for orientation) is equal to three for even n and is equal to two for odd n.

4. SOLUTION OF HOMOLOGICAL EQUATIONS

4.1. Proof of Theorem 1. The vector θ is tangential to a symplectic leaf of the first bracket at the point L if it lies in the range of the operator of the first Hamiltonian structure, i.e., can be expressed in the form $\theta = (LX - XL)_+$ for a certain X. Our problem is to describe the vectors θ for which this equation on X, called the homological equation, is solvable. For $X = \sum a_j(x) D^{-j}$ and $\theta = \sum \theta_l D^l$ it has the explicit form: $\theta_{n-k} D^{n-k} = (a'_{k-1} + Q_{k-1}(u, a)) D^{n-k}$, where $2 \leqslant k \leqslant n$, and the expressions $Q_m(u, a)$ contain unknown functions $a_j(x)$ with the indices $j \leqslant m - 1$ and their derivatives [here the functions $a_j(x)$ are themselves factors of $u_i^{(l)}(x)$, $l \geqslant 1$]. Thus,

$$a'_{k-1} = \theta_{n-k} - Q_{k-1}(u, a). \qquad (*)$$

Let us observe that a necessary condition for the solvability of these equations is that $\theta_{n-1} \equiv 0$, i.e., the action of X does not change the coefficient $u_{n-1}(x)$, and if this coefficient is not equal to zero (case of GL), then $u_{n-1}(x)$ is an invariant of a leaf and the leaves themselves have infinite codimension.

Now let $\theta_{n-1} \equiv 0$. Then Eqs. (*) are solved recurrently, starting from a_1. Indeed, knowing preceding a_i, we find a'_{i+1}. The existence of a solution $a_{i+1} \in C^\infty(S^1)$ is equivalent to the single condition $\int_{S^1} (\theta_{n-i-2} - Q_{i+1}(u, a))\, dx = 0$, which is a condition on the mean $\int_{S^1} \theta_{n-i-2}$. Equations (*) define a_{i+1} up to the addition of a constant. It can be chosen arbitrarily, since the mean of the function a_j does not influence the condition for the solvability of the following equations by virtue of the condition on Q:

$$\int_{S^1} a_j(x)\, u^{(l)}(x)\, dx = \int_{S^1} (a_j(x) + c)\, u^{(l)}(x)\, dx.$$

Thus, in the case of SL, the tangent plane to a leaf of the first bracket is defined by n − 1 conditions on the means of the magnitude of the functions θ_i. In the cases of Sp and SO, these conditions are compatible with the requirement of (skew) self-adjointness and analogous computations give k conditions on the mean. Thus, the codimension of the tangent plane to a leaf is constant on $\mathfrak{L}$, and the plane itself depends smoothly on the point $L \in \mathfrak{L}$. The conditions on θ_i are found recurrently, starting from θ_{n-2}, from the coefficients of the operator L, and, therefore, the surface of the leaf is the graph of a suitable mapping, which proves that the bundle on the leaves is trivial and the leaves are contractible.

4.2. Proof of Theorem 2. LEMMA 1 (see [21]). The action of the symbol X on the solution ψ of the equation $L\psi = 0$, induced by the second Hamiltonian structure, is given by the equation $\psi_X = -(XL)_+\psi$.

Proof. Let L_ε be a deformation of the operator L ($L_0 = L$) such that $L_X^2 = dL_\varepsilon/d\varepsilon|_{\varepsilon=0}$, and ψ_ε be the solution of the equation $L_\varepsilon\psi_\varepsilon = 0$, $\psi_0 = \psi$. By definition, $\psi_X = d\psi_\varepsilon/d\varepsilon|_{\varepsilon=0}$. It follows from $dL_\varepsilon\psi_\varepsilon/d\varepsilon|_{\varepsilon=0} = 0$ that $L_X^2\psi = L\psi_X = 0$. Using the explicit form of L_X^2, we get

$$-(LX)_+L\psi + L(XL)_+\psi + L\psi_X = 0,$$

and, by virtue of $L\psi = 0$, the first term is equal to zero. Thus,

$$L(\psi_X + (XL)_+\psi) = 0,$$

whence for an arbitrary solution ψ we have $\psi_X = -(XL)_+\psi + \varphi$, where φ is a solution of the equation $L\psi = 0$. The action of the symbol X does not depend on the choice of the term φ. Therefore, setting it equal to zero, we get the action of X on the solutions that is compatible with the action of X on L. The lemma is proved.

It is obvious that this action (by virtue of linearity with respect to ψ) preserves the monodromy operator: $(\psi + \varepsilon\psi_X)(x + 2\pi) = (M\psi - \varepsilon(XL)_+M\psi)(x) = M(\psi + \varepsilon\psi_X)(x)$. Now let L_ε be an isomonodromic deformation of the operator L. We show that L_ε belongs to the same symplectic leaf as L.

The deformation of the fundamental system of solutions $\Psi_\varepsilon = \Psi + \varepsilon\Phi + O(\varepsilon^2)$, corresponding to L_ε, can be chosen such that its monodromy matrix does not depend on ε.

Fundamental Lemma. The homological equation $\Psi_X = \Phi$ has a unique solution – the symbol X with periodic coefficients.

Proof. For $X = \sum a_j(x) D^{-j}$ the equation $\Psi_X = \Phi$ takes the "triangular form" $\sum [a_i(x) + P_i(u, a)]\Psi^{(n-i)} = \Phi$, where $\Psi^{(j)} = D^j\Psi$ and the expressions $P_i(u, a)$ contain unknown functions $a_j(x)$ with the indices $j < i$, $P_1 = 0$. The resulting system of linear equations is solved by Cramer's rule: $a_i + P_i = \det \Delta_i(x)/W(\Psi(x))$, where $W(\Psi(x)) = \det V(\Psi(x))$ is the Wronskian, the determinant of the Wronski matrix V of the system of linear equations and Δ_i is the matrix obtained from $V(\Psi(x))$ by replacing the column $\Psi^{(n-i)}$ by Φ. Now $a_i(x)$ are computed recurrently, starting from a_1.

Under displacement by 2π, the Wronski matrix V and also Δ_i are multiplied on the left by the monodromy matrix M (here we have used the fact that Ψ and Φ have the same monodromy), and, therefore, the ratio $\det\Delta_i/\det V$ is a periodic function, i.e., $a_i(x) \in C^\infty(S^1, \mathbf{k})$. The lemma is proved.

Under restrictions on $\mathfrak{L}$, for each of the matrix groups SL, Sp, and SO the above-found solution X, by definition, satisfies the condition $D^n + L_X \in \mathfrak{L}$, since the deformation belongs to $\mathfrak{L}$. Theorem 2 is proved.

Theorem 3 also follows immediately from the unique explicit solution of the homological equation, since, when the equation depends smoothly on the parameter, the solutions also depend smoothly on it [$W(\Psi(x))$ vanishes nowhere].

Remark. The geometrical meaning of the solution of the homological equation is as follows: To the fundamental system of solutions Ψ there corresponds a quasiperiodic nondegenerate curve γ and to its deformation Φ there corresponds an infinitesimal deformation of the curve. Under the action of the monomial $X_k = a_k(x)D^{-k}$, the curve is deformed along F_{n-k} – the (n – k)-th subspace of the accompanying flag, and the projection of the rate of deformation on F_{n-k}/F_{n-k-1}.

Proof of Proposition 1. Let $t \to \sum_{l=1}^{n}\psi_i(t)e_i \in \mathbf{R}^n$ be the solution curve of the equation $L\psi = 0$ such that $W(\Psi(t)) = 1$. We show that the coordinates $\eta_j(t)$ of the dual curve

$$t \to \sum_{j=1}^{n} \eta_j(t)e_j^* = \begin{vmatrix} e_1^* & \ldots & e_n^* \\ \psi_1(t) & \ldots & \psi_n(t) \\ \ldots & \ldots & \ldots \\ \psi_1^{(n-2)}(t) & \ldots & \psi_n^{(n-2)}(t) \end{vmatrix}.$$

satisfy the equation $L^*\eta = 0$ [this is precisely the curve of the osculating planes and $(e_1^*, \ldots, e_n^*)$ is the basis in $\mathbf{R}^{n*}$)]. We prove this assertion for $\eta_1(t) = |\psi, \ldots, \psi^{(n-2)}|$, where $|\psi, \ldots, \psi^{(n-2)}|$ denotes the determinant of the matrix, formed from the columns $\psi^{(i)} := (\psi_2^{(i)}(t), \ldots, \psi_n^{(i)}(t))^t$.

Let $L = D^n + \sum_{i=0}^{n-2} u_i D^i$. Then $D|\psi, \ldots, \hat{\psi}^{(i)}, \ldots, \psi^{(n-1)}| = |\psi, \ldots, \hat{\psi}^{(i-1)}, \ldots, \psi^{(n-1)}| + (-1)^{n-i+1}u_i\eta_1$. Indeed,

$$D|\psi, \ldots, \hat{\psi}^{(i)}, \ldots, \psi^{(n-1)})| = |\psi, \ldots, \hat{\psi}^{(i-1)}, \ldots, \psi^{(n-1)}| + |\psi, \ldots, \hat{\psi}^{(i)}, \ldots, \hat{\psi}^{(n-1)}, \psi^{(n)}|,$$

but, by virtue of $\psi^{(n)} = -\sum_{i=0}^{n-2} u_i\psi^{(i)}$,

$$|\psi, \ldots, \hat{\psi}^{(i)}, \ldots, \hat{\psi}^{(n-1)}, \psi^{(n)}| = \left|\psi, \ldots, \hat{\psi}^{(i)}, \ldots, \psi^{(n-2)}, -\sum_{i=1}^{n-2} u_i\psi^{(i)}\right| =$$
$$= (-1)^{n-i+1}u_i(t)|\psi, \ldots, \psi^{(n-2)}| = (-1)^{n-i+1}u_i(t)\eta_1.$$

Now $D^n\eta_1(t) = D \cdot \ldots \cdot D|\psi, \ldots, \psi^{(n-2)}| = -\sum D^i(-1)^{n-i}u_i(t)\eta_1(t)$. ■

5. CONCLUDING REMARKS AND UNSOLVED PROBLEMS

5.1. The topological structure of the symplectic leaves of the second Gel'fand–Dikii bracket remains an important and practically uninvestigated problem. In distinction from the contiguity of leaves, which is completely described by the monodromy operator, the monodromies as well as the homotopy type of the corresponding quasiperiodic nondegenerate curve determine the topology. For example, in the case of $SL(2, \mathbf{R})$ the traces (on the transversals) of the leaves with different monodromies can be contractible as well as homotopically equivalent to the circle (see Fig. 1). Moreover, different leaves with the same monotromy operator have different topologies depending on the homotopy type of the quasiperiodic nondegenerate curve that distinguishes them.

Let us observe that for $SL(3, \mathbf{R})$ the symplectic leaf, corresponding to nonoscillatory quasiperiodic nondegenerate curves on $\mathbf{RP}^2$ with the identity monodromy (see Fig. 2a), is contractible. Indeed, the set of closed nondegenerate nonoscillatory curves can be retracted onto the set of curves of constant (chosen beforehand) curvature. These curves form the sphere S^2 (with the curve we can associate its center of gravity) and all are equivalent to each other, and, therefore, correspond to the same differential operator.

5.2. For a complete classification of leaves it is necessary to solve the problem of the number of the homotopy classes of nondegenerate curves in $\mathbf{RP}^n$ with given monodromy. The classes of closed curves have been described with the proof of B. Z. Shapiro's conjecture. The problem with an arbitrary monodromy is at present solved only for the case of $\mathbf{RP}^2$ [22].

5.3. In [15] the Gel'fand–Dikii brackets, associated with the Lie groups SO(2k), were introduced. In this case the space $\mathfrak{L}$ is a two-sheeted covering of the set of the skew-symmetric pseudodifferential operators L, satisfying the conditions $\operatorname{ord} L = 2k - 1$ and $L_- = fD^{-1}$, where $f \in C^\infty(S^1)$.

On the other hand, the Gel'fand–Dikii brackets are introduced on the gauge classes of matrix differential operators and, in this connection, the only continuous invariant of a leaf is the monodromy operator [11]. It would be interesting to find direct proof of Theorems 2 and 3 for the groups SO(2k) in the terminology of pseudodifferential operators and, in particular, to find the analogue of monodromy for these operators.

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