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Modules of Differential Operators on the Real Line

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The space $\mathscr{D}^k$ of kth-order linear differential operators on $\mathbb{R}$ is equipped with a natural two-parameter family of structures of Diff($\mathbb{R}$)-modules. To specify this family, one considers the action of differential operators on tensor densities. We give a classification of these modules.

1. Modules of differential operators. Let $\mathscr{D}^k$ be the space of kth-order linear differential operators

$$A = A_k(x)\partial^k + A_{k-1}(x)\partial^{k-1} + \cdots + A_0(x), \qquad \partial = \frac{d}{dx},\ A_i(x) \in C^\infty(\mathbb{R}), \tag{1}$$

on $\mathbb{R}$ (or S^1). There exists a natural two-parameter family of structures of Diff($\mathbb{R}$)- (and Vect($\mathbb{R}$)-)modules on $\mathscr{D}^k$, where Diff($\mathbb{R}$) is the group of diffeomorphisms of $\mathbb{R}$ and Vect($\mathbb{R}$) is the Lie algebra of vector fields on $\mathbb{R}$. To define these structures, we assume that differential operators act on tensor densities on $\mathbb{R}$; namely, $A\colon \mathscr{F}_\lambda \to \mathscr{F}_\mu$, where $\mathscr{F}_\lambda$ is the Diff($\mathbb{R}$)-module of tensor densities of degree $-\lambda$, $\lambda \in \mathbb{R}$, that is, densities of the form $\varphi(x)(dx)^{-\lambda}$ (see [6]). Let $\mathscr{D}^k_{\lambda,\mu}$ be the Diff($\mathbb{R}$)- (and Vect($\mathbb{R}$)-)module of kth-order differential operators acting from $\mathscr{F}_\lambda$ to $\mathscr{F}_\mu$. The action of Vect($\mathbb{R}$) on $\mathscr{D}^k_{\lambda,\mu}$ is given by the formula

$$L^{\lambda,\mu}_X(A) = L^\mu_X \circ A - A \circ L^\lambda_X, \qquad X = X(x)\partial, \tag{2}$$

where $L^\lambda_X = X\partial - \lambda X'$ is the operator of Lie derivative on $\mathscr{F}_\lambda$ and $X'(x) = dX(x)/dx$.

The module $\mathscr{D}_{\lambda,\mu}$ of all differential operators acting from $\mathscr{F}_\lambda$ to $\mathscr{F}_\mu$ has the natural filtration $\mathscr{D}^0_{\lambda,\mu} \subset \mathscr{D}^1_{\lambda,\mu} \subset \cdots \subset \mathscr{D}^k_{\lambda,\mu} \subset \cdots$.

The aim of the present paper is to classify these modules. We shall give a complete list of isomorphisms between distinct modules $\mathscr{D}^k_{\lambda,\mu}$.

The classification problem for modules of differential operators on a smooth manifold was posed (for $\lambda = \mu$) and solved for modules of second-order operators in [3]. The modules $\mathscr{D}^k_{\lambda,\lambda}$ on $\mathbb{R}$ (or S^1) were classified in [8]. In the multidimensional case, this classification problem was recently solved in [12, 16]. The one-dimensional case proves to be exceptional, and here the results are more interesting.

2. Preliminary remarks. First, note that the difference $\delta = \mu - \lambda$ of weights is an invariant: the condition $\mathscr{D}^k_{\lambda,\mu} \cong \mathscr{D}^k_{\lambda',\mu'}$ implies that $\mu - \lambda = \mu' - \lambda'$. This is a consequence of the equivariance with respect to the vector field $x\partial$.

Recall that the passage to the *adjoint* differential operator defines an isomorphism $*\colon \mathscr{D}^k_{\lambda,\mu} \xrightarrow{\cong} \mathscr{D}^k_{-1-\mu,-1-\lambda}$ of Vect($\mathbb{R}$)-modules. The module on the right-hand side is called the *adjoint module* of $\mathscr{D}^k_{\lambda,\mu}$. A module with $\lambda + \mu = -1$ is said to be *self-adjoint*.

3. Classification results. Let us now give a complete classification of the modules $\mathscr{D}^k_{\lambda,\mu}$.

Theorem 1. *For $k \leqslant 3$, all* Diff($\mathbb{R}$)*-modules $\mathscr{D}^k_{\lambda,\mu}$ with given k and $\delta = \mu - \lambda$ are isomorphic except for the modules listed in the following table and the corresponding adjoint modules $\mathscr{D}^k_{-1-\mu,-1-\lambda}$:*

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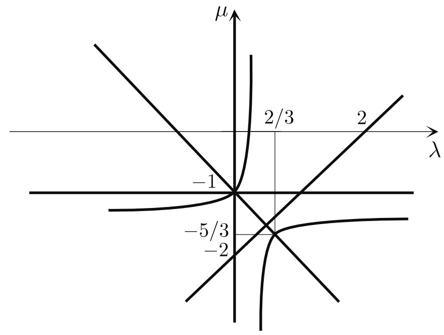

Fig. 1

k	1	2	3
(λ,μ)	$(0,-1)$	$(0,\mu)$ $(1/2,-3/2)$	$(0,\mu)$ $(\lambda,-1-\lambda)$ $(\lambda,(4\lambda-1)/(-3\lambda+1))$ $(\lambda,\lambda-2)$

(3)

We say that a module $\mathscr{D}^k_{\lambda,\mu}$ is *singular* if it is not isomorphic to any other module except for the adjoint module.

Theorem 2. *All modules listed in table* (3) *are singular.*

Figure 1 shows the singular modules $\mathscr{D}^3_{\lambda,\mu}$.

Note that the line $\mu=-1$ corresponds to the adjoint modules of the modules with $\lambda=0$ in the table.

Remark. The best-known classical example of modules of differential operators is the module of Sturm–Liouville operators $A=\partial^2+u(x)$ on S^1 acting from $\mathscr{F}_{1/2}$ to $\mathscr{F}_{-3/2}$. This is a submodule of the self-adjoint singular module $\mathscr{D}^2_{1/2,-3/2}$. Note that this module is related to the Virasoro algebra [10]. We hope that the other singular modules defined above also have some interesting interpretation.

Now let us state the most general result.

Theorem 3. *For $k\geqslant 4$, there are no isomorphisms between distinct* $\mathrm{Diff}(\mathbb{R})$*-modules $\mathscr{D}^k_{\lambda,\mu}$ except for the passage to the adjoint module.*

We note that the counterpart of Theorem 3 in the multidimensional case holds for $k\geqslant 3$ (see [12, 16]).

4. Modules of symbols. The space of *symbols* of differential operators (1) is isomorphic to the space $\mathrm{Pol}^k(T^*\mathbb{R})$ of all functions on $T^*\mathbb{R}$ polynomial (of order $\leqslant k$) in the fibers. Being treated as a $\mathrm{Diff}(\mathbb{R})$- (and $\mathrm{Vect}(\mathbb{R})$-)module, the symbol space corresponding to $\mathscr{D}_{\lambda,\mu}$ has the form

$$\mathrm{Pol}(T^*\mathbb{R})\otimes\mathscr{F}_\delta\cong\mathscr{F}_\delta\oplus\mathscr{F}_{\delta+1}\oplus\cdots\oplus\mathscr{F}_{\delta+k}\oplus\cdots,\qquad \delta=\mu-\lambda. \tag{4}$$

For brevity, we denote $\mathrm{Pol}^k(T^*\mathbb{R})\otimes\mathscr{F}_\delta$ by $\mathscr{S}^k_\delta$ and $\mathrm{Pol}(T^*\mathbb{R})\otimes\mathscr{F}_\delta$ by $\mathscr{S}_\delta$.

The module of symbols is isomorphic to the graded module associated with the filtered module of differential operators: $\mathscr{S}_\delta=\mathrm{gr}\,\mathscr{D}_{\lambda,\mu}$.

5. Modules of differential operators over sl(2, ℝ). Consider the Lie algebra $\mathrm{sl}(2,\mathbb{R})\subset\mathrm{Vect}(\mathbb{R})$ with basis $\{\partial,x\partial,x^2\partial\}$ and the restriction of the action (2) to it. This Lie algebra plays a special role and allows one to identify $\mathscr{D}^k_{\lambda,\mu}$ with $\mathscr{S}^k_\delta$, where $\delta=\mu-\lambda$, in a canonical way.

The following result shows that for generic values of δ the $\mathrm{sl}(2,\mathbb{R})$-module $\mathscr{D}^k_{\lambda,\mu}$ is isomorphic to $\mathscr{S}^k_\delta$.

Theorem 4. (i) *If $\delta\neq -1,-3/2,-2,-5/2,\ldots,-k$, then $\mathscr{D}^k_{\lambda,\mu}$ is isomorphic to $\mathscr{S}^k_\delta$ as an* $\mathrm{sl}(2,\mathbb{R})$*-module.*

(ii) *There exists a unique isomorphism* $\sigma\colon \mathscr{D}^k_{\lambda,\mu} \to \mathscr{S}^k_\delta$ *of* $\mathrm{sl}(2,\mathbb{R})$*-modules preserving the principal symbol.*

The isomorphism σ is called the *projectively equivariant symbol mapping*. We need an explicit formula borrowed from [2] (see also [11, 13] for the multidimensional case). The mapping σ takes each homogeneous kth-order differential operator $A = A_k(x)\partial^k$ to the polynomial $\sigma(A) = \sum_{i=0}^k \alpha_i^k A_k^{(i)} \xi^i$, where $A_k^{(i)} = \partial^i A_k$ and the constants α_i^k are given by

$$\alpha_i^k = \frac{\binom{k}{i}\binom{2\lambda-k+i}{2\lambda-k}}{\binom{2\delta+2k-i+1}{2\delta+2k-2i+1}}\,. \tag{5}$$

The singular values of δ at which the denominator in (5) is zero are called *resonances*. In this case, the Casimir operator of $\mathrm{sl}(2,\mathbb{R})$ on the module $\mathscr{D}_{\lambda,\mu}$ has multiple eigenvalues (see [4]).

Let us now state the result for the resonant case.

Proposition 1. *If* $\delta = -1, -3/2, -2, -5/2, \dots, -k$*, then the desired isomorphism* σ *of* $\mathrm{sl}(2,\mathbb{R})$*-modules exists if and only if*

$$(\lambda,\mu) = \begin{cases} (\ell/2, -(\ell+2)/2), & 0 \leqslant \ell \leqslant k-1,\\ (\ell/2, -(\ell+1)/2) \ \textit{or}\ ((\ell-1)/2, -(\ell+1)/2), & 1 \leqslant \ell \leqslant k-1. \end{cases}$$

However, in this case the isomorphism of $\mathrm{sl}(2,\mathbb{R})$-modules is no longer unique.

The main idea of proof of Theorems 1 and 3 is to use the $\mathrm{sl}(2,\mathbb{R})$-equivariant symbol mapping to reduce the action of $\mathrm{Vect}(\mathbb{R})$ on $\mathscr{D}_{\lambda,\mu}$ to a canonical form. In other words, we shall use the diagram

$$\begin{array}{ccc} \mathscr{D}_{\lambda,\mu} & \xrightarrow{L_X^{\lambda,\mu}} & \mathscr{D}_{\lambda,\mu} \\ \sigma\big\downarrow & & \big\downarrow\sigma \\ \mathscr{S}_\delta & \xrightarrow{\sigma\circ L_X^{\lambda,\mu}\circ\sigma^{-1}} & \mathscr{S}_\delta \end{array} \tag{6}$$

and compare the action $\sigma \circ L_X^{\lambda,\mu} \circ \sigma^{-1}$ with the standard action of $\mathrm{Vect}(\mathbb{R})$ on $\mathscr{S}_\delta$.

6. The action of Vect($\mathbb{R}$) in sl(2,$\mathbb{R}$)-invariant form. Let us compute the action of $\mathrm{Vect}(\mathbb{R})$ on $\mathscr{D}^k_{\lambda,\mu}$ for $k \leqslant 4$ in terms of the $\mathrm{sl}(2,\mathbb{R})$-equivariant symbol.

A straightforward computation results in the following formula. The action $\sigma \circ L_X^{\lambda,\mu} \circ \sigma^{-1}$ of $\mathrm{Vect}(\mathbb{R})$ on the space of fourth-order polynomials $P = P_4\xi^4 + \dots + P_0$ has the form $\sigma \circ L_X^{\lambda,\mu} \circ \sigma^{-1}(P) = P_4^X\xi^4 + \dots + P_0^X$ with

$$\begin{aligned} P_4^X &= L_X^{\delta+4}(P_4),\\ P_3^X &= L_X^{\delta+3}(P_3),\\ P_2^X &= L_X^{\delta+2}(P_2) + \beta_2^4 J_3(X,P_4),\\ P_1^X &= L_X^{\delta+1}(P_1) + \beta_1^3 J_3(X,P_3) + \beta_1^4 J_4(X,P_4),\\ P_0^X &= L_X^{\delta}(P_0) + \beta_0^2 J_3(X,P_2) + \beta_0^3 J_4(X,P_3) + \beta_0^4 J_5(X,P_4), \end{aligned} \tag{7}$$

where the β_i^j are some constants (see formula (9) below) and the bilinear mappings $J_\ell(X,P_k)$ are so-called *transvectants*, i.e., the unique (up to a constant) $\mathrm{sl}(2,\mathbb{R})$-equivariant bilinear mappings on tensor densities (see [9] and references therein).

In our case,

$$\begin{aligned} J_3(X,P_s) &= X'''P_s,\\ J_4(X,P_s) &= sX^{(IV)}P_s + 2X'''P_s',\\ J_5(X,P_s) &= s(2s-1)X^{(V)}P_s + 5(2s-1)X^{(IV)}P_s' + 10X'''P_s'' \end{aligned} \tag{8}$$

for $s = 2, 3, 4$.

Note that the higher-order coefficients P_4 and P_3 are transformed as tensor densities; the additional terms arising in the transformation of lower-order coefficients distinguish the action $\sigma \circ L_X^{\lambda,\mu} \circ \sigma^{-1}$ from the standard action of $\mathrm{Vect}(\mathbb{R})$ on $\mathscr{S}_\delta$.

The numerical coefficients β_i^j in (7) are given by

$$
\begin{aligned}
\beta_2^4(\lambda,\mu) &= \frac{(6\lambda-4)\delta+6\lambda^2+6\lambda-5}{7+2\delta},\\
\beta_1^3(\lambda,\mu) &= \frac{(3\lambda-1)\delta+3\lambda^2+3\lambda-1}{5+2\delta},\\
\beta_1^4(\lambda,\mu) &= \frac{(\delta+2\lambda+1)[(4\lambda-1)\delta+4\lambda(\lambda+1)]}{(2+\delta)(3+\delta)(4+\delta)},\\
\beta_0^2(\lambda,\mu) &= \frac{\lambda(\delta+\lambda+1)}{3+2\delta},\\
\beta_0^3(\lambda,\mu) &= \frac{\lambda(\delta+\lambda+1)(\delta+2\lambda+1)}{(1+\delta)(2+\delta)(3+\delta)},\\
\beta_0^4(\lambda,\mu) &= \frac{\lambda(\delta+\lambda+1)(4\delta^2+12\lambda\delta+12\delta+12\lambda^2+12\lambda+11)}{(1+\delta)(3+2\delta)(5+2\delta)(7+2\delta)(4+\delta)}.
\end{aligned}
\tag{9}
$$

Note that the resonant values of δ in Theorem 4 and Proposition 1 are just the ones for which the coefficients (9) are not defined.

7. The construction of isomorphisms. Here we outline the proofs of Theorems 1 and 3. Details and computations will be published elsewhere.

To prove Theorem 1, let us construct the desired isomorphism explicitly in terms of the projectively equivariant symbol. We set $P_4 \equiv 0$ and consider the linear mapping $T\colon \mathscr{D}^3_{\lambda,\mu} \to \mathscr{D}^3_{\lambda',\mu'}$ defined by

$$
T(P_3\xi^3+\dots+P_0) = P_3\xi^3 + \frac{\beta'^3_0\beta^2_0}{\beta^3_0\beta'^2_0}\,P_2\xi^2 + \frac{\beta'^3_1}{\beta^3_1}\,P_1\xi^1 + \frac{\beta'^3_0}{\beta^3_0}\,P_0,
\tag{10}
$$

where the $\beta'^j_i = \beta^j_i(\lambda',\mu')$ are the coefficients of the action (7). This mapping proves to be an isomorphism of $\mathrm{Vect}(\mathbb{R})$-modules. Indeed, one readily checks that this mapping commutes with the action of the Lie algebra $\mathrm{Vect}(\mathbb{R})$ on $\mathscr{D}^3_{\lambda,\mu}$ and $\mathscr{D}^3_{\lambda',\mu'}$. This proves Theorem 1 for nonresonant values of δ (i.e., for $\delta \neq -1, -3/2, -2, -5/2, -3$), since we have used the projectively equivariant symbol in the construction of the isomorphism.

The isomorphism (10), however, makes sense even for resonant δ, which completes the proof of Theorem 1. To check this, one can rewrite formula (10) in terms of the coefficients of differential operators (1). We omit this computation, which is straightforward.

Let us now consider an isomorphism $T\colon \mathscr{D}^k_{\lambda,\mu} \to \mathscr{D}^k_{\lambda',\mu'}$ with $k \geqslant 4$. Since T is an isomorphism of $\mathrm{Vect}(\mathbb{R})$-modules, it is also an isomorphism of $\mathrm{sl}(2,\mathbb{R})$-modules. The uniqueness of the $\mathrm{sl}(2,\mathbb{R})$-equivariant symbol mapping shows that the linear mapping $\sigma \circ T \circ \sigma^{-1}$ on $\mathscr{S}^k_\delta$ is diagonal and is given by multiplication by a constant on each homogeneous component (as is the isomorphism (10) above).

One can readily show (see [12]) that the restriction of T to $\mathscr{D}^4_{\lambda,\mu} \subset \mathscr{D}^k_{\lambda,\mu}$ must be an isomorphism onto the submodule $\mathscr{D}^4_{\lambda',\mu'}$. Again, assuming that δ is nonresonant, so that there exists a projectively equivariant symbol, one checks that the linear mapping $T(P_4\xi^4+\dots+P_0) = P_4\xi^4+\tau_3P_3\xi^3+\dots+\tau_0P_0$ with indeterminate τ_i depending on λ, μ, λ', and μ' intertwines the actions (7) of $\mathrm{Vect}(\mathbb{R})$ on $\mathscr{D}^4_{\lambda,\mu}$ and $\mathscr{D}^4_{\lambda',\mu'}$ if and only if

$$
\begin{aligned}
&\tau_4\beta'^4_2 = \tau_2\beta^4_2, \quad \tau_4\beta'^4_1 = \tau_1\beta^4_1, \quad \tau_4\beta'^4_0 = \tau_0\beta^4_0,\\
&\tau_3\beta'^3_1 = \tau_1\beta^3_1, \quad \tau_3\beta'^3_0 = \tau_0\beta^3_0, \quad \tau_2\beta'^2_0 = \tau_0\beta^2_0.
\end{aligned}
$$

One can readily check that this system has solutions only if $\lambda = \lambda'$ or $\lambda + \mu' = -1$. The first isomorphism is tautological, and the second isomorphism is just the passage to the adjoint module. This proves Theorem 3 for nonresonant δ.

We omit the computations in the resonant case.

8. The cohomology of Vect(ℝ) related to the modules $\mathscr{D}^k_{\lambda,\mu}$. It is a general fact that a filtered module V over a Lie algebra $\mathfrak{g}$ can be viewed as a *deformation* of the corresponding graded module $\operatorname{gr} V$. This deformation is related to the first cohomology group with coefficients in $\operatorname{End}(\operatorname{gr} V)$. We refer here to the classical theory of Richardson–Nijenhuis [17, 18].

The module $\mathscr{D}_{\lambda,\mu}$ is therefore a nontrivial deformation of the symbol module $\mathscr{S}_\delta$ with $\mu - \lambda = \delta$ (see [3, 8, 14]). This module is related to the cohomology space

$$H^1(\mathrm{Vect}(\mathbb{R}); \mathrm{End}(\mathscr{S}_\delta)) = \bigoplus_{k>\ell} H^1(\mathrm{Vect}(\mathbb{R}); \mathrm{Hom}(\mathscr{F}_{\delta+k}, \mathscr{F}_{\delta+\ell})).$$

More precisely, the transvectants $J_\ell(X, P_s)$ with $\ell = 3, 4, 5$ define *nontrivial* 1*-cocycles* on $\mathrm{Vect}(\mathbb{R})$ with values in $\mathrm{Hom}(\mathscr{F}_{\delta+s}, \mathscr{F}_{\delta+s-\ell+1})$ by the formula

$$C_\ell(X) = J_\ell(X, \,\cdot\,) \tag{11}$$

(see [8]). The action (7) of $\mathrm{Vect}(\mathbb{R})$ is a nontrivial deformation of the action on the space of symbols.

Remark. The cohomology space $H^1(\mathrm{Vect}(\mathbb{R}); \mathrm{Hom}(\mathscr{F}_\nu, \mathscr{F}_{\nu'}))$ was defined in [5] for the Lie algebra of polynomial vector fields on $\mathbb{R}$; see also [15] for the case of polynomial vector fields on S^1. For the Lie algebra of smooth vector fields, the corresponding space was described in [14] (see also [1]).

Let us use the nontrivial cocycles (11) to prove Theorem 2. The fact that these 1-cocycles are nontrivial implies the existence of singular modules whenever at least one of the coefficients β_i^j in (7) is zero. Theorem 2 now follows from the explicit formulas (9).

Remark. Note that the cocycles $C_\ell(X)$ with $\ell = 3, 4, 5$ vanish on $\mathrm{sl}(2, \mathbb{R})$ and define nontrivial classes in the *relative cohomology* $H^1(\mathrm{Vect}(S^1), \mathrm{sl}(2, \mathbb{R}); \mathrm{Hom}(\mathscr{F}_\lambda, \mathscr{F}_\mu))$.

9. Obstructions to the existence of an sl(2, ℝ)-equivariant symbol mapping. For the resonant values $\delta = -1, -3/2, -2, -5/2, \ldots, -k$, there exist a series of cohomology classes of $\mathrm{sl}(2, \mathbb{R})$ that are obstructions to the existence of the isomorphism in Theorem 4. More precisely, consider the linear mappings $C_2^n \colon \mathrm{sl}(2, \mathbb{R}) \to \mathrm{Hom}(\mathscr{F}_{n/2}, \mathscr{F}_{-1-n/2})$ given by

$$C_2^n(X)(a(dx)^{-n/2}) := X'' a^{(n)} (dx)^{1+n/2}. \tag{12}$$

One can check (see [7]) that these mappings are nontrivial 1-cocycles on $\mathrm{sl}(2, \mathbb{R})$ for every $n \in \mathbb{N}$. These cocycles arise in the action (2) of $\mathrm{sl}(2, \mathbb{R})$ on $\mathscr{D}_{\lambda,\mu}$.

One can nevertheless define a canonical symbol mapping in the resonant case such that its deviation from $\mathrm{sl}(2, \mathbb{R})$-equivariance is measured by the corresponding cocycle (12). Then the proofs of Theorems 1 and 3 can be obtained for the resonant case in the same way as for the generic case.

Remark. (a) The cohomology of $\mathrm{sl}(n+1, \mathbb{R})$ with coefficients in $\mathrm{Hom}(\mathscr{S}^k_\delta, \mathscr{S}^\ell_\delta)$ was computed in [11]. Multidimensional analogs of Theorem 4 and Proposition 1 were also obtained in this paper.

(b) The quotient modules $\mathscr{D}^k_{\lambda,\lambda} / \mathscr{D}^\ell_{\lambda,\lambda}$ with $k > \ell$ were classified in [13].

(c) The relative cohomology space $H^1(\mathrm{Vect}(S^1), \mathrm{sl}(2, \mathbb{R}); \mathscr{D}_{\lambda,\mu})$ (i.e., the cohomology of the complex of cocycles on $\mathrm{Vect}(\mathbb{R})$ vanishing on $\mathrm{sl}(2, \mathbb{R})$) was first considered in [8] and completely described in [1, 14].

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