

DUAL NUMBERS, WEIGHTED QUIVERS, AND EXTENDED SOMOS AND GALE-ROBINSON SEQUENCES

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ABSTRACT. We investigate a general method that allows one to construct new integer sequences extending existing ones. We apply this method to the classic Somos-4 and Somos-5, and the Gale-Robinson sequences, as well as to more general class of sequences introduced by Fordy and Marsh, and produce a great number of new sequences. The method is based on the notion of “weighted quiver”, a quiver with a \mathbb{Z} -valued function on the set of vertices that obeys very special rules of mutation.

1. INTRODUCTION

A *dual number* is a pair of real numbers a and b , written in the form $a + b\varepsilon$, subject to the condition that $\varepsilon^2 = 0$. Dual numbers form a commutative algebra, they were introduced by Clifford in 1873, and since then have found applications in geometry and mathematical physics. For example, according to E. Study, the space of oriented lines in \mathbb{R}^3 is the unit sphere in the 3-dimensional space over dual numbers [15]. Surprisingly, dual numbers are not frequent guests in number theory and combinatorics. In this paper, we will use dual numbers to construct a large family of integer sequences.

Let $(a_n)_{n \in \mathbb{N}}$ be an integer sequence defined by some recurrence and initial conditions. We will consider a pair of sequences, $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$, organized as a sequence of dual numbers:

$$(1) \quad A_n := a_n + b_n \varepsilon.$$

We assume that $(A_n)_{n \in \mathbb{N}}$ satisfies either exactly the same recurrence as $(a_n)_{n \in \mathbb{N}}$, or its certain deformation. We furthermore choose the same initial conditions for $(a_n)_{n \in \mathbb{N}}$ and arbitrary initial conditions for $(b_n)_{n \in \mathbb{N}}$. We show that, in many interesting cases, the sequence $(b_n)_{n \in \mathbb{N}}$, defined in this way, is an integer sequence. This method was suggested in [13], and tested on the Somos-4 sequence, producing new integer sequences.

1.1. Somos, Gale-Robinson and beyond. Let us give a brief overview of the integer sequences that we will consider.

The Somos-4 and Somos-5 sequences are the sequences of integers, $(a_n)_{n \in \mathbb{N}}$, defined by the recurrences:

$$a_{n+4}a_n = a_{n+3}a_{n+1} + a_{n+2}^2 \quad \text{and} \quad a_{n+5}a_n = a_{n+4}a_{n+1} + a_{n+3}a_{n+2},$$

and the initial conditions: $a_1 = a_2 = a_3 = a_4 = 1$ and $a_1 = a_2 = a_3 = a_4 = a_5 = 1$, respectively. These sequences were discovered by Michael Somos in 80's, and they are discrete analogs of elliptic functions. Their integrality, observed and conjectured by Somos, was later proved by several authors; for a historic account see [7].

A more general class of integer sequences generalizing the Somos sequences, called the three term Gale-Robinson sequences, are defined by the recurrences:

$$(2) \quad a_{n+N}a_n = a_{n+N-r}a_{n+r} + a_{n+N-s}a_{n+s},$$

where $1 \leq r < s \leq \frac{N}{2}$, and the initial conditions $(a_1, \dots, a_N) = (1, \dots, 1)$. Their integrality was proved in [4]. A combinatorial proof was then given in [1] and [14], a proof that explicitly uses quiver mutations and cluster algebras was presented in [6].

A large class of integer sequences generalizing those of Gale-Robinson was introduced in [6] (see also [10]). The recurrence is of the general form

$$(3) \quad a_{n+N}a_n = P(a_{n+N-1}, \dots, a_{n+1}),$$

where the polynomial P is a sum of two monomials: $P = P_1 + P_2$. The initial conditions are those of Gale-Robinson. A simple example of such sequences is:

$$a_{n+4}a_n = a_{n+3}^p a_{n+1}^p + a_{n+2}^q,$$

with arbitrary positive integers p, q . Note that this particular sequence was already considered in [7], where their integrality was claimed.

The method of [6] is based on the Fomin-Zelevinsky *Laurent phenomenon* [4] and on the notion of *period 1 quiver*, i.e., a quiver that rotates under mutations.

1.2. Extensions. The same arguments show that, for $A_n = a_n + b_n \varepsilon$ satisfying the recurrence

$$A_{n+N}A_n = P(A_{n+N-1}, \dots, A_{n+1}),$$

where P is as in [6], and $(a_1, \dots, a_N) = (1, \dots, 1)$, the sequence $(b_n)_{n \in \mathbb{N}}$ is integer for an arbitrary choice of integral initial conditions (b_1, \dots, b_N) . In fact, this is a direct consequence of the classic Laurent phenomenon.

Note that the recurrence for $(b_n)_{n \in \mathbb{N}}$ obtained in this way, is nothing other than the *linearization* of (3). This linearization procedure already provides a large number of new integer sequences.

We will also consider the recurrences of the following form:

$$(4) \quad A_{n+N}A_n = P_1 + P_2(1 + w\varepsilon) \quad \text{and} \quad A_{n+4}A_n = P_1(1 + w\varepsilon) + P_2,$$

where w is an arbitrary integer, and P_i stands for $P_i(A_{n+N-1}, \dots, A_{n+1})$, $i = 1, 2$. In this case, $(b_n)_{n \in \mathbb{N}}$ satisfies a *non-linear* recurrence. More precisely, the recurrence for $(b_n)_{n \in \mathbb{N}}$ is given by an affine function with polynomial in $(a_n)_{n \in \mathbb{N}}$ coefficients.

We show, in particular, that these non-linear extensions of the Gale-Robinson sequences are always integer. We give a sufficient condition for the corresponding period 1 quiver that guaranties that the extensions of the form (4) generates an integer sequence (b_n) . Integrality of the sequences defined by (4) is a consequence of a version of the Laurent phenomenon proved in [13] for cluster superalgebras.

1.3. Weighted quivers. Our main tool is what we call a weighted quiver. This is a usual quiver \mathcal{Q} (without 1- or 2-cycles), together with a function $w : \mathcal{Q}_0 \rightarrow \mathbb{Z}$ on the set of vertices. Quiver mutations for such weight functions are defined as follows. Label the vertices by positive integers $1, \dots, n$, the mutation μ_k at k th vertex sends w to the new function $\mu_k(w)$ defined by:

$$\begin{aligned} \mu_k(w)(i) &= w(i) + [b_{ki}]_+ w(k), \quad i \neq k, \\ \mu_k(w)(k) &= -w(k), \end{aligned}$$

where $[b_{ki}]_+$ is the number of arrows from the vertex k to the vertex i , and if the vertices are oriented from i to k , then $[b_{ki}]_+ = 0$. The exchange relations are also modified.

Let us mention that the mutation rule of the weight function that we use (but not the exchange relations), have already been introduced by several authors; see formula (2.3) of [9] and [8],[2]. It would be interesting to investigate the relations of our work with these papers.

We classify the period 1 quivers (in the sense of Fordy-Marsh [6]) that have a period 1 weight function.

2. EXTENSIONS OF THE SOMOS-4 SEQUENCE

We start with the extensions of the Somos-4 sequence, briefly considered in [13]. The goal of this section is to show that the new sequences arising in this way have nice properties. This section is based on a computer program written by Michael Somos, to whom we are most grateful. It can be considered as a motivation for the rest of the paper.

2.1. Linearization. Consider first the recurrence

$$(5) \quad A_{n+4}A_n = A_{n+3}A_{n+1} + A_{n+2}^2,$$

where $A_n = a_n + b_n\varepsilon$ as in (1). The sequence $(a_n)_{n \in \mathbb{N}}$ is then the Somos-4 sequence:

$$a_n = 1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, 8209, 83313, 620297, 7869898, \dots$$

while the sequence $(b_n)_{n \in \mathbb{N}}$ satisfies:

$$b_{n+4}a_n + b_na_{n+4} = a_{n+1}b_{n+3} + 2a_{n+2}b_{n+2} + a_{n+3}b_{n+1},$$

which is the *linearization* of the Somos-4 recurrence.

The space of solutions of the linearized system is a four-dimensional vector space. Every sequence $(b_n)_{n \in \mathbb{N}}$ satisfying this recurrence is a linear combination of the sequences with one of the following initial conditions:

$$(b_1, b_2, b_3, b_4) = (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), \text{ or } (0, 0, 0, 1).$$

Let us denote these sequences by $(b_n^1)_{n \in \mathbb{N}}, (b_n^2)_{n \in \mathbb{N}}, (b_n^3)_{n \in \mathbb{N}}, (b_n^4)_{n \in \mathbb{N}}$, respectively.

We have the following properties:

- (1) The four sequences $(b_n^1)_{n \in \mathbb{N}}, (b_n^2)_{n \in \mathbb{N}}, (b_n^3)_{n \in \mathbb{N}}, (b_n^4)_{n \in \mathbb{N}}$ are integer.
- (2) The Somos-4 sequence is their sum, i.e.,

$$a_n = b_n^1 + b_n^2 + b_n^3 + b_n^4,$$

for every n .

Integrality of each of the four sequences $(b_n^i)_{n \in \mathbb{N}}$ follows from the classic Laurent phenomenon. Indeed, A_n with $n \geq 5$ is a Laurent polynomial in A_1, A_2, A_3, A_4 . However, the denominators of this Laurent polynomial are monomials in a_1, a_2, a_3, a_4 , while b_1, b_2, b_3, b_4 enter (linearly) into the numerators. Indeed, since $\varepsilon^2 = 0$, one has

$$\frac{1}{a + b\varepsilon} = \frac{1}{a} - \frac{b}{a^2}\varepsilon.$$

Property (2) follows from the fact that the sequences $(b_n)_{n \in \mathbb{N}}$ satisfying the recurrence (5) form a vector space, and the initial condition of the sum $(b_n^1 + b_n^2 + b_n^3 + b_n^4)_{n \in \mathbb{N}}$ is precisely $(1, 1, 1, 1)$, i.e., that of $(a_n)_{n \in \mathbb{N}}$. Property (2) means that the sequences $(b_n^1)_{n \in \mathbb{N}}, (b_n^2)_{n \in \mathbb{N}}, (b_n^3)_{n \in \mathbb{N}}, (b_n^4)_{n \in \mathbb{N}}$ provide a canonical way to decompose the Somos-4 sequence with respect to the initial conditions.

Note, however, that the sequences $(b_n^1)_{n \in \mathbb{N}}$ and $(b_n^2)_{n \in \mathbb{N}}$ are not positive, the first values being:

$$\begin{array}{l} b_n^1 = 1, 0, 0, 0, -2, -2, -10, -46, -103, -933, -4681, -27912, -375536, \dots \\ b_n^2 = 0, 1, 0, 0, 1, -2, 2, -1, -40, 140, -696, -265, 38478, \dots \\ b_n^3 = 0, 0, 1, 0, 2, 4, 5, 48, 94, 635, 4732, 18594, 299835, \dots \\ b_n^4 = 0, 0, 0, 1, 1, 3, 10, 22, 108, 472, 2174, 17792, 120536, \dots \end{array}$$

It would be interesting to understand the properties of these sequences. Computer calculations show that $b_n^1 < 0$, for every $n \leq 100$. The sequence $(b_n^2)_{n \in \mathbb{N}}$ becomes positive for $n \geq 15$ (this checked for $n \leq 100$). Conjecturally, the sequence $(b_n^3)_{n \in \mathbb{N}}$ is positive. The sequence $(b_n^4)_{n \in \mathbb{N}}$ is positive for $n \leq 25$, however, $b_n^4 < 0$, for $26 \leq n \leq 100$.

2.2. Two non-linear extensions. Consider now the extensions of the Somos-4 sequence, satisfying the recurrences:

$$A_{n+4}A_n = A_{n+3}A_{n+1} + A_{n+2}^2(1 + w\varepsilon)$$

$$A_{n+4}A_n = A_{n+3}A_{n+1}(1 + w\varepsilon) + A_{n+2}^2,$$

where $w \in \mathbb{Z}$ is an arbitrary (fixed) integer. The corresponding recurrences for $(b_n)_{n \in \mathbb{N}}$ are non-linear:

$$(6) \quad b_{n+4}a_n = a_{n+1}b_{n+3} + 2a_{n+2}b_{n+2} + a_{n+3}b_{n+1} - b_na_{n+4} + wa_{n+2}^2,$$

$$(7) \quad b_{n+4}a_n = a_{n+1}b_{n+3} + 2a_{n+2}b_{n+2} + a_{n+3}b_{n+1} - b_na_{n+4} + wa_{n+3}a_{n+1},$$

respectively, where $(a_n)_{n \in \mathbb{N}}$ is the initial Somos-4 sequence. It was proved in [13] that, for any choice of integral initial conditions, the sequence $(b_n)_{n \in \mathbb{N}}$ satisfying (6) or (7) is integer.

The choice of zero initial conditions:

$$(8) \quad (b_1, b_2, b_3, b_4) = (0, 0, 0, 0)$$

is now the most natural one. Indeed, any solution (b_n) of each of the recurrences (6) and (7) is the sum of the solution with the initial condition (8) and a solution of the linear recurrence (5). In other words, the solutions of (6) and (7) form an affine subspace.

Choosing $w = 1$, the sequences $(b_n)_{n \in \mathbb{N}}$ defined by the recurrences (6) and (7) and zero initial conditions (8) start as follows:

$$b_n = 0, 0, 0, 0, 1, 2, 10, 48, 160, 1273, 7346, 51394, 645078, 5477318, 87284761 \dots$$

$$b_n = 0, 0, 0, 0, 1, 3, 10, 59, 198, 1387, 9389, 57983, 752301, 6851887, 97297759 \dots$$

respectively. We conjecture the positivity of these sequences, and we checked it for $n \leq 100$. Furthermore, conjecturally, both of the above sequences grow faster than $(a_n)_{n \in \mathbb{N}}$.

3. WEIGHTED QUIVERS, MUTATIONS AND EXCHANGE RELATIONS

In this section, we introduce the notion of weighted quiver. We then describe the mutation rules of such objects, extending the usual mutation rules of quivers. We also describe the modified exchange relations for weighted quivers, and formulate the corresponding Laurent phenomenon.

The notion of weighted quiver is equivalent to the ‘‘simplified version’’ of cluster superalgebra with two odd variables [13].

3.1. Mutation rules. A quiver \mathcal{Q} is an oriented finite graph with vertex set \mathcal{Q}_0 and the set of arrows \mathcal{Q}_1 . Usually, the vertices of \mathcal{Q} will be labeled by the letters $\{x_1, \dots, x_N\}$, where $N = |\mathcal{Q}_0|$, considered as formal variables.

In [3], Fomin and Zelevinsky defined the rules of mutation of a quiver, under the assumption that \mathcal{Q} has no 1-cycles and 2-cycles. The structure of \mathcal{Q} can then be represented as an $N \times N$ -skew-symmetric matrix (b_{ij}) , where b_{ij} is the number of arrows between the vertices x_i and x_j . Note that the sign of b_{ij} depends on the orientation: $b_{ij} > 0$ if the arrows are oriented from x_i to x_j and negative otherwise.

The mutation of the quiver $\mu_k : \mathcal{Q} \rightarrow \mathcal{Q}'$ at vertex x_k is defined by the following three rules:

- for every path $(x_i \rightarrow x_k \rightarrow x_j)$ in \mathcal{Q} , add an arrow $(x_i \rightarrow x_j)$;
- reverse all the arrows incident with x_k ;
- remove all 2-cycles created by the first rule.

Definition 3.1. We call a weighted quiver a quiver \mathcal{Q} with a function

$$w : \mathcal{Q}_0 \rightarrow \mathbb{Z}.$$

The function w associates to every variable x_i its weight, $w_i := w(x_i)$.

The mutation $\mu_k(w)$ of the weight function w is performed according to the following two rules:

- (1) for every arrow $(x_k \rightarrow x_i)$, change the value w_i to $w_i + w_k$. In other words, the new weight function $\mu_k(w)$ is defined by

$$\mu_k(w)(x_i) := w_i + [b_{ki}]_+ w_k,$$

where

$$[b_{ki}]_+ = \begin{cases} b_{ki}, & \text{if } b_{ki} \geq 0, \\ 0, & \text{otherwise} \end{cases}$$

for all $i \neq k$;

- (2) reverse the sign of w_k , i.e.,

$$\mu_k(w)(x_k) := -w_k.$$

3.2. Exchange relations. Recall that the mutation μ_k of the quiver \mathcal{Q} replaces the variable x_k by the new function x'_k defined by the formula:

$$x_k x'_k = \prod_{x_k \rightarrow x_j} x_j + \prod_{x_i \rightarrow x_k} x_i,$$

where the products are taken over the set of arrows $(x_i \rightarrow x_k) \in \mathcal{Q}_1$ and $(x_k \rightarrow x_j) \in \mathcal{Q}_1$, respectively (with fixed k). The above formula is called the *exchange relation*. The *Laurent phenomenon*, proved in [3], states that every (rational) function obtained by a series of mutations is actually a Laurent polynomial in the initial variables $\{x_1, \dots, x_N\}$.

Given a weighted quiver (\mathcal{Q}, w) , we assume that the vertices are labeled by the variables $\{X_1, \dots, X_N\}$ written as dual numbers:

$$X_i = x_i + y_i \varepsilon,$$

where x_i and y_i are the usual commuting variables. The exchange relations are defined as follows.

Definition 3.2. The mutation μ_k of (\mathcal{Q}, w) replaces the variable X_k by a new variable, X'_k , defined by the formula

$$(9) \quad X_k X'_k = \prod_{X_k \rightarrow X_j} X_j + (1 + w_k \varepsilon) \prod_{X_i \rightarrow X_k} X_i;$$

the other variables remain unchanged.

This is a particular case of the exchange relations defined in [13].

3.3. Laurent phenomenon. The following version of Laurent phenomenon is proved in [13] (this is the simplest case of Theorem 1).

Theorem 3.3. For every weighted quiver (\mathcal{Q}, w) , all the functions X'_k, X''_k, \dots , obtained recurrently by any series of consecutive mutations, are Laurent polynomials in the initial coordinates $\{X_1, \dots, X_N\}$.

Remark 3.4. Note that Laurentness in $\{X_1, \dots, X_N\}$ means that the denominators are monomials in the variables $\{x_1, \dots, x_N\}$, while the variables $\{y_1, \dots, y_N\}$ enter (linearly) into the numerators.

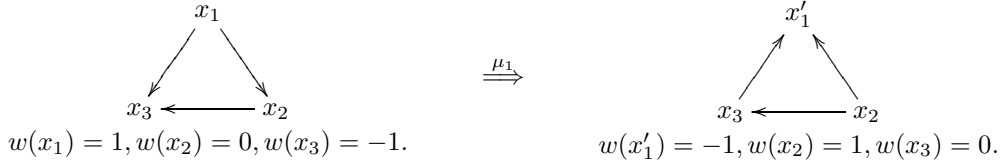
4. PPERIOD 1 WEIGHTED QUIVERS

Period 1 quivers were introduced and classified in [6]. These are quivers for which there exists a vertex such that the quiver rotates under the mutation at this vertex. More precisely, a period 1 quiver remains unchanged after the mutation composed with the shift of the indices of the vertices $i \rightarrow i - 1$.

In this section, we answer the question which period 1 quivers have period 1 weight functions. A period 1 quiver equipped with a period 1 weight function guarantees the integrality of sequence $(b_n)_{n \in \mathbb{N}}$ defined by (4).

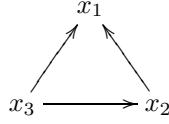
4.1. **Examples.** We start with simple examples.

Example 4.1. a) Consider the following weighted quiver with three vertices. After mutation at x_1 the quiver rotates, together with the weight function:



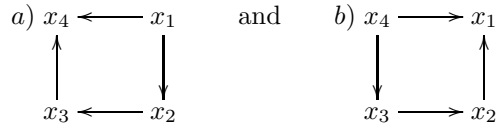
We will say in such a situation, that the *weight function has period 1*.

b) On the other hand, for the quiver with the inverted orientation:



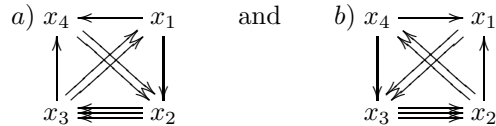
which is also of period 1, there is no weight function of period 1.

Example 4.2. Similarly, for the following quivers of period 1 with four vertices and the opposite orientations:



the weight function $w(x_1) = 1, w(x_2) = 0, w(x_3) = 0, w(x_4) = -1$, has period 1 in the first case, and there is no such function in the second case.

Example 4.3. Another interesting example is the following Somos-4 quivers (cf. [6] and [10]):



Period 1 weight function exists for both choices of the orientation:

$$\begin{array}{cccc}
 w(x_1) = 1, & w(x_2) = 0, & w(x_3) = 0, & w(x_4) = -1, \\
 w(x_1) = 1, & w(x_2) = 1, & w(x_3) = -1, & w(x_4) = -1,
 \end{array}$$

respectively.

4.2. Period 1 weight functions: criterion of existence. Period 1 quivers were classified in [6]. The *primitive quiver* $P_N^{(t)}$, where $1 \leq t \leq \frac{N}{2}$, is a quiver with n vertices and n arrows, such that every vertex x_i is joined with the vertex $x_{i+t} \pmod{N}$ where the indices are always taken in the set $\{1, \dots, N\}$. The arrow is oriented from the vertex with the greater label to the vertex with the smaller label. One then has for $P_N^{(t)}$:

$$b_{ij} = \begin{cases} -1, & j - i = t, \\ 1, & i - j = t, \\ 0, & \text{else.} \end{cases}$$

For instance, the quiver considered in Examples 4.1 b) and 4.2 b) are the quivers $P_3^{(1)}$ and $P_4^{(1)}$, respectively.

Given a quiver \mathcal{Q} , the opposite quiver $-\mathcal{Q}$ is obtained by reversing the orientation. For example, the quivers in Examples 4.1 a) and 4.2 a) are the quivers $-P_3^{(1)}$ and $-P_4^{(1)}$, respectively. More generally, if $c \in \mathbb{Z}$, the quiver $c\mathcal{Q}$ is obtained by multiplying the number of arrows between every two vertices, x_i and x_j by c . Finally, a sum of two quivers is obtained by superposition of their arrows.

It was proved in [6], that every period 1 quiver can be obtained as a linear combination of so-called primitive quivers and a correction term. More precisely, let c_1, \dots, c_r be arbitrary integers, where $r = \lfloor \frac{N}{2} \rfloor$. A period 1 quiver is of the form:

$$(10) \quad \mathcal{Q} = c_1 P_N^{(1)} + \dots + c_r P_N^{(r)} + \mathcal{Q}',$$

where \mathcal{Q}' is a quiver with the vertices x_2, \dots, x_N . Since we will only consider the mutation at x_1 , this ‘‘correcting term’’ \mathcal{Q}' will not change the exchange relations. We thus omit the explicit form of \mathcal{Q}' ; see [6] and [10].

Let us use the notation $[c]_- = \begin{cases} c, & c \leq 0 \\ 0, & \text{otherwise.} \end{cases}$

Theorem 4.4. *Given a period 1 quiver \mathcal{Q} , there exists a period 1 weight function on \mathcal{Q} if and only if*

$$(11) \quad \begin{aligned} [c_1]_- + \dots + [c_r]_- &= 1, & \text{if } N \text{ is odd} \\ 2[c_1]_- + \dots + 2[c_{r-1}]_- + [c_r]_- &= 2, & \text{if } N \text{ is even.} \end{aligned}$$

The period 1 weight function is unique up to an integer multiple.

Proof. Assume that a period 1 quiver \mathcal{Q} admits a period 1 weight function w . By definition 3.1, the mutation at the first vertex, μ_1 , transforms the weight function as follows:

$$\begin{aligned} w_1 &\mapsto -w_1, \\ w_i &\mapsto w_i + [b_{1i}]_+ w_1, \end{aligned}$$

for all $1 \leq i \leq N$. Since w is of period 1, this implies the following system of linear equations:

$$\begin{aligned} w_n &= -w_1, \\ w_1 &= w_2 + [b_{12}]_+ w_1, \\ w_2 &= w_3 + [b_{13}]_+ w_1, \\ &\dots \\ w_{N-1} &= w_n + [b_{1N}]_+ w_1, \end{aligned}$$

that has (a unique) solution if and only if the following condition is satisfied:

$$[b_{12}]_+ + \dots + [b_{1N}]_+ = 2.$$

Finally, from (10), one has $[b_{1i}]_+ = [c_{i-1}]_-$, if $i \leq r$ and $[b_{1i}]_+ = [c_{N-i+1}]_-$, if $i \geq r$. The above necessary and sufficient condition for the existence of the function w then coincides with (11). \square

5. APPLICATIONS TO INTEGER SEQUENCES

We apply the above constructions to integer sequences.

5.1. The general method. Given a weighted quiver (\mathcal{Q}, w) of period 1, by Theorem 3.3, performing an infinite series of consecutive mutations: μ_1, μ_2, \dots , one obtains a sequence of Laurent polynomials $(X_n)_{n \in \mathbb{N}}$ in the initial variables $\{X_1, \dots, X_N\}$. This sequence satisfies the recurrence:

$$X_{n+N}X_n = \prod_{1 \leq i \leq N-1} X_{n+i}^{[b_{1i}]_+} (1 + w_1 \varepsilon) + \prod_{1 \leq i \leq N-1} X_{n+i}^{[b_{1i}]_-}.$$

Recall that $X_i = x_i + y_i \varepsilon$. Choosing the initial conditions $(x_1, \dots, x_N) := (1, \dots, 1)$ and arbitrary integers $(y_1, \dots, y_N) := (b_1, \dots, b_N)$, one obtains a sequence $(A_n)_{n \in \mathbb{N}}$, where $A_n = a_n + b_n \varepsilon$. The constructed integer sequence $(b_n)_{n \in \mathbb{N}}$ is the desired extension of $(a_n)_{n \in \mathbb{N}}$.

Let us give further examples.

5.2. Sequences of order 2. We illustrate the idea of substitution of dual numbers into recurrences on a very simple classic example.

Consider the classic Fibonacci numbers $(F_n) = 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \dots$ and let us split (F_n) into two subsequences:

$$a_n := F_{2n-1}, \quad \tilde{a}_n := F_{2n}.$$

Both of them satisfy quadratic recurrences that differ by a sign:

$$(12) \quad a_{n+2}a_n = a_{n+1}^2 + 1, \quad \tilde{a}_{n+2}\tilde{a}_n = \tilde{a}_{n+1}^2 - 1.$$

The above recurrences are known as ‘‘Cassini’s identity’’. The initial conditions for these sequences are: $(a_0, a_1) = (1, 1)$ and $(\tilde{a}_0, \tilde{a}_1) = (0, 1)$.

We will consider the sequences of dual numbers:

$$A_n := a_n + b_n \varepsilon, \quad \tilde{A}_n := \tilde{a}_n + \tilde{b}_n \varepsilon,$$

with the recurrence relations generalizing (12).

Linearization: dual Fibonacci and Lucas numbers. Suppose that (A_n) and (\tilde{A}_n) satisfy the similar recurrences:

$$(13) \quad A_{n+2}A_n = A_{n+1}^2 + 1, \quad \tilde{A}_{n+2}\tilde{A}_n = \tilde{A}_{n+1}^2 - 1.$$

Equivalently, the sequences (a_n) and (\tilde{a}_n) are as above, and (b_n) and (\tilde{b}_n) are defined by:

$$(14) \quad b_{n+2}a_n = 2b_{n+1}a_{n+1} - b_n a_{n+2}, \quad \tilde{b}_{n+2}\tilde{a}_n = 2\tilde{b}_{n+1}\tilde{a}_{n+1} - \tilde{b}_n \tilde{a}_{n+2}.$$

The sequence (b_n) is integer for an arbitrary choice of integral initial conditions $(b_0, b_1) = (p, q)$.

It turns out that the classic Lucas numbers naturally appear in the ‘‘dual Fibonacci’’ sequences.

Proposition 5.1. *The sequences of odd (reps. even) Lucas numbers:*

$$b_n = L_{2n-1}, \quad \tilde{b}_n = L_{2n},$$

satisfy the recurrence (13).

Proof. Using the explicit formulas for the Fibonacci and Lucas numbers

$$F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}, \quad L_n = \varphi^n + (-\varphi)^{-n},$$

where φ is the golden ratio, one checks directly that the recurrences (14) are satisfied. \square

The Lucas solutions to (13) start as follows:

n	0	1	2	3	4	5	6	7	...	n	0	1	2	3	4	5	6	7	...
a_n	1	1	2	5	13	34	89	233	...	\tilde{a}_n	0	1	3	8	21	55	144	377	...
b_n	-1	1	4	11	29	76	199	521	...	\tilde{b}_n	2	3	7	18	47	123	322	843	...

Furthermore, one has a 2-parameter family of solutions to the recurrence (13):

n	0	1	2	3	4	5	6	7	...
a_n	1	1	2	5	13	34	89	233	...
b_n	$-p$	q	$2p + 2q$	$8p + 3q$	$27p + 2q$	$86p - 10q$	$265p - 66q$	$798p - 277q$...

The situation is more complicated for the sequence $(\tilde{b}_n)_{n \in \mathbb{Z}}$. Arbitrary initial conditions $(\tilde{b}_0, \tilde{b}_1)$ do not lead to an integer sequence. But one obtains a two-parameter family of integer sequences by choosing the initial conditions $(\tilde{b}_1, \tilde{b}_2) = (3p, q)$ with arbitrary p and q .

n	1	2	3	4	5	6	7	...
\tilde{a}_n	1	3	8	21	55	144	377	...
\tilde{b}_n	$3p$	q	$6q - 24p$	$25q - 128p$	$90q - 507p$	$300q - 1778p$	$954q - 5835p$...

Note that the sequence of coefficients of q is A001871.

“Limping” Fibonacci sequence. The classic odd Fibonacci sequence $a_n = F_{2n-1}$ satisfies the first recurrence in (12). It can be generated by consecutive mutations $\mu_0, \mu_1, \mu_0, \mu_1, \dots$ of the quiver with two vertices and two arrows:

$$x_0 \leftarrow x_1,$$

called the 2-Kronecker quiver. Clearly, there is no period 1 weight function w , but the function $w \equiv 1$ has period 4.

The sequence of consecutive mutations at vertices x_1, x_2, x_1, \dots then leads to the recurrence

$$(15) \quad A_{n+2}A_n = A_{n+1}^2 \left(1 + (-1)^{\frac{(n+1)(n+2)}{2}} \varepsilon \right) + 1.$$

More precisely, $(b_n)_{n \in \mathbb{Z}}$ satisfies

$$b_{n+2}a_n + b_n a_{n+2} = 2b_{n+1}a_{n+1} + (-1)^{\frac{(n+1)(n+2)}{2}} a_{n+1}^2.$$

Let us consider the initial conditions $a_0 = a_1 = 1$ and $b_0 = b_1 = 0$.

It turns out that the sequence $(b_n)_{n \in \mathbb{N}}$ also consists of Fibonacci numbers, but this time with even indices, and taken in a surprising order:

n	0	1	2	3	4	5	6	7	...
a_n	1	1	2	5	13	34	89	233	...
b_n	0	0	1	8	21	21	55	377	...

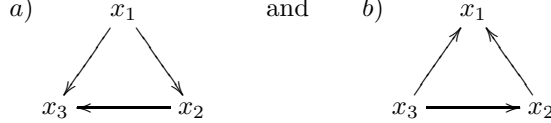
More precisely,

$$b_n = \begin{cases} F_{2n}, & n \equiv 0, 3 \pmod{4}, \\ F_{2n-2}, & n \equiv 1, 2 \pmod{4}. \end{cases}$$

5.3. Sequences of order 3. Consider the sequence A005246 satisfying the recurrence ¹

$$a_{n+3}a_n = a_{n+2}a_{n+1} + 1,$$

and starting as follows: $a_n = 1, 1, 1, 2, 3, 7, 11, 26, 41, 97, 153, 362, 571, 1351, \dots$ This sequence can be generated by the period 1 quivers



already considered in Example 4.1.

By Theorem 4.4, the first quiver has a period 1 weight function, but not the second one. Therefore, the sequence $(b_n)_{n \in \mathbb{N}}$ defined by the recurrence

$$A_{n+3}A_n = A_{n+2}A_{n+1} + 1 + w \varepsilon$$

and arbitrary integer initial conditions is integer. Our results do not give any information about the sequence $(b_n)_{n \in \mathbb{N}}$ satisfying

$$A_{n+3}A_n = A_{n+2}A_{n+1} (1 + \varepsilon) + 1,$$

but numerical experiments show that it is not integer.

However, consider again the quiver b). The weight function $w(x_1) = w(x_2) = w(x_3) = 1$ is of period 6. Indeed, after three consecutive mutations $\mu_3 \circ \mu_2 \circ \mu_1$, the function w changes its sign and becomes $w(x'_1) = w(x'_2) = w(x'_3) = -1$, while the quiver remains unchanged. Therefore, the recurrence

$$(16) \quad A_{n+3}A_n = A_{n+2}A_{n+1} \left(1 + (-1)^{\frac{(n+1)(n+2)(n+3)}{6}} \varepsilon \right) + 1$$

defines integer sequences $(b_n)_{n \in \mathbb{N}}$. Note that the exponent is chosen to obtain the sign sequence $+, +, +, -, -, -, +, +, +, \dots$. Written more explicitly, $(b_n)_{n \in \mathbb{N}}$ satisfies the non-linear recurrence

$$b_{n+3}a_n = b_{n+2}a_{n+1} + b_{n+1}a_{n+2} - b_n a_{n+3} + (-1)^{\frac{(n+1)(n+2)(n+3)}{6}} a_{n+2}a_{n+1}.$$

For example, zero initial conditions lead to the following sequence

$$b_n = 0, 0, 0, 1, 3, 15, 17, 43, 2, 112, 84, \dots$$

5.4. Non-homogeneous Somos-4 sequence. Let p and q be positive integers, and consider the sequence $(a_n)_{n \in \mathbb{N}}$ defined by the recurrence

$$a_{n+4}a_n = a_{n+3}^p a_{n+1}^p + a_{n+2}^q$$

and the initial conditions $a_0 = a_1 = a_2 = a_3 = 1$. This sequence was considered in [7]; see also [6].

Corollary 5.2. (i) The sequence $(b_n)_{n \in \mathbb{N}}$, defined by the recurrence

$$(17) \quad A_{n+4}A_n = A_{n+3}A_{n+1} + A_{n+2}^q (1 + \varepsilon)$$

with arbitrary integer initial conditions (b_1, b_2, b_3, b_4) , is integer.

(ii) The sequence $(b_n)_{n \in \mathbb{N}}$, defined by the recurrence

$$(18) \quad A_{n+4}A_n = A_{n+3}^p A_{n+1}^p (1 + \varepsilon) + A_{n+2}$$

with arbitrary integer initial conditions (b_1, b_2, b_3, b_4) , is integer.

¹Note that, unlike the Somos sequences, this sequence also satisfies a linear recurrence: $a_{n+4} = 4a_{n+2} - a_n$.

Proof. Following [6], consider the quivers (that differ only by orientation):

$$(19) \quad \begin{array}{ccc} x_4 & \xleftarrow{p} & x_1 \\ & \searrow^q & \nearrow^q \\ x_3 & \xleftarrow{p(q+1)} & x_2 \end{array} \quad \text{and} \quad \begin{array}{ccc} x_4 & \xrightarrow{p} & x_1 \\ & \searrow^q & \nearrow^q \\ x_3 & \xrightarrow{p(q+1)} & x_2 \end{array}$$

where the labels p, q and $p(q+1)$ stand for the number of arrows. Each of them rotates under the series of consecutive mutations $\mu_1, \mu_2, \mu_3, \dots$. For instance,

$$\begin{array}{ccc} x_4 & \xleftarrow{p} & x_1 \\ & \searrow^q & \nearrow^q \\ x_3 & \xleftarrow{p(q+1)} & x_2 \end{array} \xrightarrow{\mu_1} \begin{array}{ccc} x_4 & \xrightarrow{p} & x'_1 \\ & \searrow^q & \nearrow^q \\ x_3 & \xleftarrow{p} & x_2 \end{array}$$

This is straightforward from the definition of quiver mutations (and similarly for the twin quiver).

By Theorem 4.4, the first of the quivers (19) has a weight function of period 1, if (and only if) $p = 1$, while the second quiver has a weight function of period 1, if (and only if) $q = 2$. \square

Note that our results do not imply the converse statement. However, we conjecture that the sequence $(b_n)_{n \in \mathbb{N}}$, defined by the recurrence $A_{n+4}A_n = A_{n+3}^p A_{n+1}^p + A_{n+2}^q (1 + \varepsilon)$, is integer *if and only if* $p = 1$ (and similarly for the second case). This conjecture is confirmed by the following examples.

Example 5.3. Let us now consider the sequence $(A_n)_{n \in \mathbb{N}}$ satisfying the recurrence $A_{n+4}A_n = A_{n+3}^p A_{n+1}^p + A_{n+2}^q$, with initial conditions: $b_0 = b_1 = b_2 = b_3 = 0$, and take $q \neq 2$. Although Theorem 4.4 does not imply non-integrality of $(b_n)_{n \in \mathbb{N}}$, this sequence is not integer in all the examples we considered.

a) If $q = 0$, then the sequence starts as follows:

n	0	1	2	3	4	5	6	7	8	9	...
a_n	1	1	1	1	2	3	4	9	14	19	...
b_n	0	0	0	0	1	3	6	24	56	$\frac{307}{3}$...

b) If $q = 1$, then the sequence stops to be integer one step earlier:

n	0	1	2	3	4	5	6	7	8	...
a_n	1	1	1	1	2	3	5	13	22	...
b_n	0	0	0	0	1	3	7	32	$\frac{159}{2}$...

c) If $q = 3$, then, the sequence is:

n	0	1	2	3	4	5	6	7	8	...
a_n	1	1	1	1	2	3	11	49	739	...
b_n	0	0	0	0	1	3	18	150	$\frac{6539}{2}$...

This and many other experimental computations illustrate a sophisticated and fragile nature of the Laurent phenomenon of Theorem 3.3. It seems to occur only when there is a weighted function with period 1.

5.5. Conclusion and an open problem. The properties of the constructed integer sequences remain unexplored. In many cases, we cannot prove their positivity (although this was checked numerically for the most interesting examples), and their asymptotics are unknown.

A very interesting property of the Somos-type sequences is their relation to discrete integrable systems; see [5] and references therein. (The properties of “integrality” and “integrability” are related not only phonetically!) It will be interesting to investigate integrability of discrete dynamical systems related to the sequences constructed in this paper. For example, is the following map on \mathbb{R}^8

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \mapsto \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ (x_4x_2 + x_3^2)/x_1 \\ y_2 \\ y_3 \\ y_4 \\ (x_2y_4 + 2x_3y_3 + x_4y_2 + x_3^2)/x_1 - y_1(x_4x_2 + x_3^2)/x_1^2 \end{pmatrix}$$

that arises from the extended Somos-4 recurrence (see Section 2) completely integrable?

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