

# Deforming the Lie Algebra of Vector Fields on $S^1$ Inside the Poisson Algebra on $\dot{T}^*S^1$

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**Abstract:** We study deformations of the standard embedding of the Lie algebra  $\text{Vect}(S^1)$  of smooth vector fields on the circle, into the Lie algebra of functions on the cotangent bundle  $T^*S^1$  (with respect to the Poisson bracket). We consider two analogous but different problems: (a) formal deformations of the standard embedding of  $\text{Vect}(S^1)$  into the Lie algebra of functions on  $\dot{T}^*S^1 := T^*S^1 \setminus S^1$  which are Laurent polynomials on fibers, and (b) polynomial deformations of the  $\text{Vect}(S^1)$  subalgebra inside the Lie algebra of formal Laurent series on  $\dot{T}^*S^1$ .

## 1. Introduction

*1.1. The standard embedding.* The Lie algebra  $\text{Vect}(M)$  of vector fields on a manifold  $M$  has a natural embedding into the Poisson Lie algebra of functions on  $T^*M$ . It is defined by the standard action of the Lie algebra of vector fields on the cotangent bundle. Using the local Darboux coordinates  $(x, \xi) = (x^1, \dots, x^n, \xi_1, \dots, \xi_n)$  on  $T^*M$ , the explicit formula is:

$$\pi(X) = X\xi, \quad (1)$$

where  $X$  is a vector field:  $X = \sum_{i=1}^n X^i(x) \partial / \partial x^i$  and  $X\xi = \sum_{i=1}^n X^i(x) \xi_i$ .

The main purpose of this paper is to study deformations of the standard embedding (1).

*1.2. Deformations inside  $C^\infty(T^*M)$ .* Consider the Poisson Lie algebra of smooth functions on  $T^*M$  for an orientable manifold  $M$ . In this case, the problem of deformation of the embedding (1) has an elementary solution. The  $\text{Vect}(M)$  embedding (1) into  $C^\infty(T^*M)$  has the unique (well-known) nontrivial deformation. Indeed, given an arbitrary volume form on  $M$ , the expression:

$$\pi_\lambda(X) = X\xi + \lambda \operatorname{div} X,$$

where  $\lambda \in \mathbf{R}$ , defines an embedding of  $\operatorname{Vect}(M)$  into  $C^\infty(T^*M)$ .

The linear map:  $X \mapsto \operatorname{div} X$  is the unique nontrivial 1-cocycle on  $\operatorname{Vect}(M)$  with values in  $C^\infty(M) \subset C^\infty(T^*M)$  (cf. [2]).

*1.3. Two Poisson Lie algebras of formal symbols.* Let us consider the following two Lie algebras of Poisson on the cotangent bundle with zero section removed:  $\dot{T}^*M = T^*M \setminus M$ .

- (a) The Lie algebra  $A(M)$  of functions on  $\dot{T}^*M$  which are *Laurent polynomials* on fibers;
- (b) The Lie algebra  $\mathcal{A}(M)$  of *formal Laurent series* on  $\dot{T}^*M$ .

Lie algebras  $A(M)$  and  $\mathcal{A}(M)$  can be interpreted as *classical limits of the algebra of formal symbols* of pseudo-differential operators on  $M$ . We will show that in this case one can expect much more interesting results than those in the case of  $C^\infty(M)$ .

In both cases, the Poisson bracket is defined by the usual formula:

$$\{F, G\} = \frac{\partial F}{\partial \xi} \frac{\partial G}{\partial x} - \frac{\partial F}{\partial x} \frac{\partial G}{\partial \xi}.$$

## 2. Statement of the Problem

In this paper we will consider only the one-dimensional case:  $M = S^1$  (analogous results hold for  $M = \mathbf{R}$ ).

*2.1. Algebras  $A(S^1)$  and  $\mathcal{A}(S^1)$  in the one-dimensional case.* As vector spaces, Lie algebras  $A(S^1)$  and  $\mathcal{A}(S^1)$  have the following form:

$$A(S^1) := C^\infty(S^1) \otimes \mathbf{C}[\xi, \xi^{-1}] \quad \text{and} \quad \mathcal{A}(S^1) := C^\infty(S^1) \otimes \mathbf{C}[\xi, \xi^{-1}],$$

where  $\mathbf{C}[\xi, \xi^{-1}]$  is the space of Laurent series in one formal indeterminate.

Elements of both algebras:  $A(S^1)$  and  $\mathcal{A}(S^1)$  can be written in the following form:

$$F(x, \xi, \xi^{-1}) = \sum_{k \in \mathbf{Z}} \xi^k f_k(x),$$

where the coefficients  $f_k(x)$  are periodic functions:  $f_k(x + 2\pi) = f_k(x)$ . In the case of algebra  $A(S^1)$ , one supposes that the coefficients  $f_k \equiv 0$ , if  $|k|$  is sufficiently large; for  $\mathcal{A}(S^1)$  the condition is:  $f_k \equiv 0$ , if  $k$  is sufficiently large.

*2.2. Formal deformations of  $\operatorname{Vect}(S^1)$  inside  $A(S^1)$ .* We will study *one-parameter formal deformations* of the standard embedding of  $\operatorname{Vect}(S^1)$  into the Lie algebra  $A(S^1)$ . That means we study linear maps

$$\pi^t : \operatorname{Vect}(S^1) \rightarrow A(S^1)[[t]]$$

to the Lie algebra of series in a formal parameter  $t$ . Such a map has the following form:

$$\pi^t = \pi + t\pi_1 + t^2\pi_2 + \dots, \tag{2}$$

where  $\pi_k : \text{Vect}(S^1) \rightarrow \mathcal{A}(S^1)$  are some linear maps, such that the formal homomorphism condition is satisfied:

$$\pi^t([X, Y]) = \{\pi^t(X), \pi^t(Y)\}.$$

The general Nijenhuis–Richardson theory of formal deformations of homomorphisms of Lie algebras will be discussed in the next section.

*2.3. Polynomial deformations of the  $\text{Vect}(S^1)$  inside  $\mathcal{A}(S^1)$ .* We classify all the *polynomial* deformations of the standard embedding (1) of  $\text{Vect}(S^1)$  into  $\mathcal{A}(S^1)$ . In other words, we consider homomorphisms of the following form:

$$\pi(c) = \pi + \sum_{k \in \mathbf{Z}} \pi_k(c) \xi^k, \quad (3)$$

where  $c = c_1, \dots, c_n \in \mathbf{R}$  (or  $\mathbf{C}$ ) are parameters of deformations, each linear map  $\pi_k(c) : \text{Vect}(S^1) \rightarrow C^\infty(S^1)$  being polynomial in  $c$ ,  $\pi_k(0) = 0$  and  $\pi_k \equiv 0$  if  $k > 0$  is sufficiently large.

#### 2.4. Motivations.

- (a) Lie algebras of functions on a symplectic manifold have nontrivial formal deformations linked with so-called *deformation quantization*. The problem considered in this paper, is original and has never been discussed in the literature. However, this problem is inspired by deformation quantization.

The geometric version of the problem, deformations (up to symplectomorphism) of zero section of the cotangent bundle  $M \subset T^*M$ , has no nontrivial solutions. Existence of nontrivial deformations in the algebraic formulation that we consider here seems to be a manifestation of “quantum anomalies”.

Note that interesting examples of deformations of Lie algebra homomorphisms related to deformation quantization can be found in [12].

- (b) Lie algebras of vector fields and Lie algebras of functions on a symplectic manifold, have both nice cohomology theories, our idea is to link them together.

Lie algebras of vector fields have various nontrivial extensions. The well-known example is the *Virasoro algebra* defined as a central extension of  $\text{Vect}(S^1)$ . A series of nontrivial extensions of  $\text{Vect}(S^1)$  by modules of tensor-densities on  $S^1$  were constructed in [8, 9]. These extensions can be obtained, using a (nonstandard) embedding of  $\text{Vect}(S^1)$  into  $C^\infty(T^*S^1)$ , by restriction of the deformation of  $C^\infty(T^*S^1)$  (see [9]).

We will show that deformations of the standard embedding relate the Virasoro algebra to extensions of Poisson algebra on  $\mathbf{T}^2$  defined by A.A. Kirillov (see [4, 11]).

- (c) The following quantum aspect of the considered problem: deformations of embeddings of  $\text{Vect}(S^1)$  into the algebra of pseudodifferential operators on  $S^1$ , will be treated in a subsequent article.

### 3. Nijenhuis–Richardson Theory

Deformations of homomorphisms of Lie algebras were first considered in [6] (see also [10]). The Nijenhuis–Richardson theory is analogous to the Gerstenhaber theory of formal deformations of associative algebras (and Lie algebras) (see [3]), related cohomological calculations are parallel. Let us outline the main results of this theory.

### 3.1. Equivalent deformations.

**Definition.** Two homomorphisms  $\pi$  and  $\pi'$  of a Lie algebra  $\mathfrak{g}$  to a Lie algebra  $\mathfrak{h}$  are equivalent (cf. [6]) if there exists an interior automorphism  $I$  of  $\mathfrak{h}$  such that  $\pi' = I\pi$ .

Let us specify this definition for the two problem formulated in Sects. 2.2 and 2.3.

- (a) Two formal deformations (2)  $\pi^t$  and  $\pi'^t$  are equivalent if there exists a linear map  $I_t : A(S^1)[[t]] \rightarrow A(S^1)[[t]]$  of the form:

$$\begin{aligned} I_t &= \exp(\text{tad}_{F_1} + t^2 \text{ad}_{F_2} + \dots) \\ &= \text{id} + \text{tad}_{F_1} + t^2(\text{ad}_{F_1}^2/2 + \text{ad}_{F_2}) + \dots, \end{aligned}$$

where  $F_i \in A(S^1)$ , such that  $\pi'^t = I_t \pi^t$ . It is natural to consider such an automorphism of  $A(S^1)[[t]]$  as interior.

- (b) An automorphism  $I(c) : \mathcal{A}(S^1) \rightarrow \mathcal{A}(S^1)$  depending on the parameters  $c = c_1, \dots, c_n$ , which is of the following form:

$$I(c) = \exp\left(\sum_{i=1}^n c_i \text{ad}_{F_i} + c_i c_j \text{ad}_{F_{i,j}} + \dots\right),$$

where  $F_i, F_{i,j}, \dots \in \mathcal{A}(S^1)$  is called interior. Two polynomial deformations  $\pi(c)$  and  $\pi'(c)$  of the standard embedding  $\text{Vect}(S^1) \hookrightarrow \mathcal{A}(S^1)$  are equivalent if there exists an interior automorphism  $I(c)$ , such that  $\pi'(c) = I(c)\pi(c)$ .

**3.2. Infinitesimal deformations.** Deformations (2) and (3), modulo second order terms in  $t$  and  $c$  respectively, are called *infinitesimal*. Infinitesimal deformations of a Lie algebra homomorphism from  $\mathfrak{g}$  into  $\mathfrak{h}$  are classified by the first cohomology group  $H^1(\mathfrak{g}; \mathfrak{h})$ ,  $\mathfrak{h}$  being a  $\mathfrak{g}$ -module through  $\pi$ .

Namely, the first order terms  $\pi_1$  in (2) and  $\left. \frac{\partial \pi(c)}{\partial c_i} \right|_{c=0}$  in (3) are *1-cocycles*. Two infinitesimal deformations are equivalent if and only if the corresponding cocycles are cohomologous.

Conversely, given a Lie algebra homomorphism  $\pi : \mathfrak{g} \rightarrow \mathfrak{h}$ , an arbitrary 1-cocycle  $\pi_1 \in Z^1(\mathfrak{g}; \mathfrak{h})$  defines an infinitesimal deformation of  $\pi$ .

**3.3. Obstructions.** The *integrability conditions* are conditions for existence of (formal or polynomial) deformation corresponding to a given infinitesimal deformation.

- (a) The obstructions for existence of a formal deformation (2) belong to the second cohomology group  $H^2(\mathfrak{g}; \mathfrak{h})$ . This follows from the so-called deformation relation (see [6]):

$$d\pi^t + (1/2)[\pi^t, \pi^t] = O, \quad (4)$$

where  $[\pi^t, \pi^t]$  is a bilinear map from  $\mathfrak{g}$  to  $\mathfrak{h}$ :

$$[\pi^t, \pi^t](x, y) := \{\pi^t(x), \pi^t(y)\} - \{\pi^t(y), \pi^t(x)\}.$$

Note that the deformation relation (4) is nothing but a rewritten formal homomorphism relation (Sect. 1.4).

Equation (4) is equivalent to a series of nonlinear equations concerning the maps  $\pi_k$ :

$$d\pi_k = -(1/2) \sum_{i+j=k} [\pi_i, \pi_j].$$

The right hand side of each equation is a 2-cocycle and the equations have solutions if and only if the corresponding cohomology classes vanish.

- (b) Analogous *necessary* conditions for existence of a polynomial deformation (3) can be easily calculated.

*3.4. Remarks. Polynomial deformations.* Deformations of algebraic structures (as associative and Lie algebras, their modules and homomorphisms) *polynomially depending on parameters* are not very well studied. There is no special version of the general theory adopted to this case and the number of known examples is small (see [1]).

Theory of polynomial deformation seems to be richer than those of formal ones. The equivalence problem for polynomial deformation has additional interesting aspects related to parameter transformations (cf. Sects. 5.4 and 5.5, formulæ (11)).

#### 4. Polynomial Deformations of the Embedding of $\text{Vect}(S^1)$ into the Lie Algebra of Formal Laurent Series on $T^*S^1$

Consider the Poisson Lie algebra  $\mathcal{A}(S^1)$ . The formula (1) defines an embedding of  $\text{Vect}(S^1)$  into this Lie algebra.

The following theorem is the main result of this paper. It gives a classification of polynomial deformations of the subalgebra  $\text{Vect}(S^1) \subset \mathcal{A}(S^1)$ .

**Theorem 1.** *Every nontrivial polynomial deformation of the standard embedding of  $\text{Vect}(S^1)$  into  $\mathcal{A}(S^1)$  is equivalent to one of a two-parameter family of deformations given by the formula:*

$$\pi^{\lambda, \mu} \left( f(x) \frac{d}{dx} \right) = f \left( x + \frac{\lambda - \mu}{\xi} \right) \xi + \mu f' \left( x + \frac{\lambda - \mu}{\xi} \right), \tag{5}$$

where  $\lambda, \mu \in \mathbf{R}$  or  $\mathbf{C}$  are parameters of the deformation; the expression in the right-hand side has to be interpreted as a formal (Laurent) series in  $\xi$ .

A complete proof of this theorem is given in Sects. 4 and 5.

The explicit formula for the deformation  $\pi^{\lambda, \mu}$  is as follows:

$$\begin{aligned} \pi^{\lambda, \mu} \left( f(x) \frac{d}{dx} \right) &= f(x)\xi + \lambda f'(x) + \left( \frac{\lambda^2}{2} - \frac{\mu^2}{2} \right) f''(x)\xi^{-1} + \dots \\ &+ \left( \frac{\mu(\lambda - \mu)^k}{k!} + \frac{(\lambda - \mu)^{k+1}}{(k + 1)!} \right) f^{(k+1)}(x)\xi^{-k} + \dots \end{aligned} \tag{5'}$$

*Remark.* The formula (5) is a result of complicated calculations which will be omitted. We do not see any *a-priori* reason for its existence.

To prove Theorem 1, we apply the Nijenhuis–Richardson theory.

The first step is to classify infinitesimal deformations. One has to calculate the first cohomology of  $\text{Vect}(S^1)$  with coefficients in  $\mathcal{A}(S^1)$ . Then, one needs the integrability condition under which an infinitesimal deformation corresponds to a polynomial one.

*4.1. Algebras  $\mathcal{A}(S^1)$  and  $A(S^1)$  as  $\text{Vect}(S^1)$ -modules.* Lie algebra  $\text{Vect}(S^1)$  is a subalgebra of  $\mathcal{A}(S^1)$ . Therefore,  $\mathcal{A}(S^1)$  is a  $\text{Vect}(S^1)$ -module.

**Definition.** Consider a 1-parameter family of  $\text{Vect}(S^1)$ -actions on  $C^\infty(S^1)$  given by

$$L_{f(x)\frac{d}{dx}}^{(\lambda)}(a(x)) = f(x)a'(x) - \lambda f'(x)a(x),$$

where  $\lambda \in \mathbf{R}$ .

Denote  $\mathcal{F}_\lambda$  the  $\text{Vect}(S^1)$ -module structure on  $C^\infty(S^1)$  defined by this action.

*Remark.* Geometrically,  $L_{f d/dx}^{(\lambda)}$  is the operator of Lie derivative on *tensor-densities* of degree  $-\lambda$ . That means:  $a = a(x)(dx)^{-\lambda}$ .

**Lemma 4.1.** (i) *The Lie algebra  $A(S^1)$  is decomposed to a direct sum of  $\text{Vect}(S^1)$ -modules:*

$$A(S^1) = \bigoplus_{m \in \mathbf{Z}} \mathcal{F}_m.$$

(ii) *The Lie algebra  $\mathcal{A}(S^1)$  has the following decomposition as a  $\text{Vect}(S^1)$ -module:*

$$\mathcal{A}(S^1) = \bigoplus_{m \geq 0} \mathcal{F}_m \oplus \prod_{m < 0} \mathcal{F}_m.$$

*Proof.* Consider the subspace of  $A(S^1)$  and  $\mathcal{A}(S^1)$  consisting of functions of degree  $m$  in  $\xi$ :  $a(x)\xi^m$ . This subspace is a  $\text{Vect}(S^1)$ -module isomorphic to  $\mathcal{F}_m$ . One has:

$$\{f(x)\xi, a(x)\xi^m\} = (fa' - mf'a)\xi^m = L_{f\frac{d}{dx}}^{(m)}(a)\xi^m.$$

Therefore, algebra of Laurent polynomials is a direct sum of  $\text{Vect}(S^1)$ -modules  $\mathcal{F}_m$ .

By definition, an element of algebra  $\mathcal{A}(S^1)$  is a formal series in  $\xi$  with a finite number of terms of positive degree.

Lemma 4.1 is proven.  $\square$

*4.2. Cohomology groups  $H^1(\text{Vect}(S^1); A(S^1))$  and  $H^1(\text{Vect}(S^1); \mathcal{A}(S^1))$ .* It follows that the  $\text{Vect}(S^1)$ -cohomology with coefficients in  $A(S^1)$  is split into a direct sum:

$$H^1(\text{Vect}(S^1); A(S^1)) = \bigoplus_{m \in \mathbf{Z}} H^1(\text{Vect}(S^1); \mathcal{F}_m).$$

These cohomology groups are well known (see [2]). They are nontrivial if and only if  $m = 0, -1, -2$  and the corresponding group of cohomology are one-dimensional. Therefore, the space of first cohomology  $H^1(\text{Vect}(S^1); A(S^1))$  is three-dimensional.

It is clear that the same result holds for  $\mathcal{A}(S^1)$ :

$$H^1(\text{Vect}(S^1); A(S^1)) = H^1(\text{Vect}(S^1); \mathcal{A}(S^1)) = \mathbf{R}^3.$$

The nontrivial cocycles generating the following cohomology groups:  $H^1(\text{Vect}(S^1); A(S^1))$  and  $H^1(\text{Vect}(S^1); \mathcal{A}(S^1))$  are

$$\begin{aligned} C_0(fd/dx) &= f', \\ C_1(fd/dx) &= f''(dx), \\ C_2(fd/dx) &= f'''(dx)^2 \end{aligned}$$

with values in  $\mathcal{F}_0, \mathcal{F}_{-1}, \mathcal{F}_{-2}$  respectively.

*4.3. Infinitesimal deformations.* It follows from the Nijenhuis–Richardson theory that the calculated cohomology group classifies infinitesimal deformations of the standard embedding of  $\text{Vect}(S^1)$  into the algebras  $A(S^1)$  and  $\mathcal{A}(S^1)$  (respectively). One obtains the following result:

**Proposition 4.2.** *Every infinitesimal deformation of the standard embedding of  $\text{Vect}(S^1)$  into  $A(S^1)$  and  $\mathcal{A}(S^1)$  is equivalent to:*

$$f(x)\partial \mapsto f\xi + c_0f' + c_1f''\xi^{-1} + c_2f'''\xi^{-2}, \tag{6}$$

where  $c_0, c_1, c_2 \in \mathbf{R}$  (or  $\mathbf{C}$ ) are parameters.

To classify polynomial deformations of the standard embedding of  $\text{Vect}(S^1)$  into  $\mathcal{A}(S^1)$ , one needs now the integrability conditions on parameters  $c_0, c_1, c_2$ .

*Remark.* We will show (cf. Sect. 6) that in the case of formal deformations of  $\text{Vect}(S^1)$  into  $A(S^1)$  the corresponding integrability conditions are completely different.

### 5. Integrability Condition

**Theorem 5.1.** (i) *An infinitesimal deformation (6) corresponds to a polynomial deformation of the standard embedding  $\text{Vect}(S^1) \hookrightarrow \mathcal{A}(S^1)$ , if and only if it satisfies the following condition:*

$$6c_0^3c_2 - 3(c_0c_1)^2 - 18c_0c_1c_2 + 8c_1^3 + 9c_2^2 = 0. \tag{7}$$

(ii) *The polynomial deformation corresponding to a given infinitesimal deformation is unique up to equivalence.*

The nonlinear relation (7) is the integrability condition for infinitesimal deformations. Given a 1-cocycle  $C \in Z^1(\text{Vect}(S^1); \mathcal{A}(S^1))$  which does not satisfy this condition, there is an obstruction for existence of a polynomial deformation.

Theorem 5.1 will be proven in the end of this section.

*Remark.* The formula (7) defines a *semi-cubic* parabola. Indeed, consider the following transformation of the parameters:

$$\begin{aligned} \tilde{c}_1 &= -2c_1 + c_0^2, \\ \tilde{c}_2 &= 3(c_2 - c_0c_1) + C_0^3. \end{aligned}$$

Then, the relation (7) is equivalent to:

$$-\tilde{c}_1^3 + \tilde{c}_2^2 = 0. \tag{7'}$$

**5.1. Homogeneous deformation.** Consider an arbitrary polynomial deformation of the standard embedding, corresponding to the infinitesimal deformation (6):

$$\pi \left( f(x) \frac{d}{dx} \right) = f\xi + c_0 f' + c_1 f'' \xi^{-1} + c_2 f''' \xi^{-2} + \sum_{k \in \mathbf{Z}} P_k(c) \pi_k(f) \xi^{-k}, \quad (8)$$

where  $c = c_1, c_2, c_3$ ,  $P_k(c)$  are polynomials of degree  $\geq 2$  and  $\pi_k : \text{Vect}(S^1) \rightarrow \mathcal{F}_{-k}$  some differentiable linear maps.

Note that since the cocycles  $C_1, C_2$  and  $C_3$  are defined by differentiable maps, an arbitrary solution of the deformation problem is also defined via differentiable maps. This follows from the Gelfand-Fuks formalism of differentiable (or local) cohomology (see [2]).

**Definition.** Let us introduce a notion of homogeneity for deformations given by differentiable maps. A polynomial deformation (8) is called homogeneous if the sum of the degree in  $\xi$  and of the order of differentiation of  $f$  in each term of the right-hand side is constant.

Since the cocycles  $C_1, C_2$  and  $C_3$  are homogeneous of order 1, every homogeneous deformation (8) corresponding to a nontrivial infinitesimal deformation, is of homogeneity 1:

$$\pi(f(x)\partial) = f\xi + c_0 f' + c_1 f'' \xi^{-1} + c_2 f''' \xi^{-2} + \sum_{k \geq 3} P_k(c) f^{(k+1)} \xi^{-k}. \quad (9)$$

**Lemma 5.2.** Every polynomial deformation (8) is equivalent to a homogeneous deformation (9).

*Proof.* It is easy to see that the homomorphism equation:  $\pi([f, g]) = \{\pi(f), \pi(g)\}$  preserves the homogeneity condition. It means that the first term in (8) (the term of the lowest degree in  $c$ ) which is not homogeneous of degree 1, must be a 1-cocycle. Such a 1-cocycle is necessarily a coboundary. Indeed, each 1-cocycle is cohomologous to a linear combination of homogeneous of order 1 cocycles:  $C_1, C_2$  and  $C_3$  (cf. Sect. 4.2).

The lemma follows now from the standard Nijenhuis–Richardson technique. One can add (or remove) a coboundary in any term of the polynomial deformation (8) to obtain an equivalent one.

Lemma 5.2 is proven.  $\square$

**5.2. Uniqueness of the homogeneous deformation.**

**Proposition 5.3.** Given an infinitesimal deformation (6), if there exists a homogeneous polynomial deformation (9) corresponding to the given one, then this homogeneous polynomial deformation is unique.

*Proof.* Substitute the formula (9) to the homomorphism equation. Put  $P_0 = c_0, P_1 = c_1, P_2 = c_2$ . Collecting the terms with  $\xi^{-k}$  (where  $k \geq 3$ ), one readily obtains the following identities for polynomials  $P_k(c)$ :

$$\begin{aligned} & P_k(c) \cdot (f g' - f' g)^{(k)} = \\ & P_k(c) \cdot (f g^{(k+1)} + (k-1) f' g^{(k)} - f^{(k+1)} g - (k-1) f^{(k)} g') \\ & + \sum_{i+j=k-1} P_i(c) P_j(c) \cdot (-i f^{(i+1)} g^{(j+2)} + j f^{(i+2)} g^{(j+1)}), \end{aligned} \quad (10)$$



for every  $f(x), g(x)$ . Each of these identities defines a system of equations of the form:  $P_k(c) = \dots$ , where  $k \geq 3$  and “ $\dots$ ” means quadratic expressions of  $P_i(c)$  with  $i < k$ . Therefore, polynomials  $P_k(c)$  with  $k \geq 3$  are uniquely defined by the constants  $c_0, c_1$  and  $c_2$ .

Proposition 5.3 is proven.  $\square$

5.3. *Integrability condition is necessary.* The first three identities (10) give the following system of equations:

$$\begin{aligned} 2P_3(c) &= 2c_0c_2 - c_1^2, \\ 5P_4(c) &= 3c_0P_3(c) - c_1c_2, \\ \begin{cases} 9P_5(c) = 4c_0P_4(c) - c_1P_3(c) \\ 5P_5(c) = 3c_1P_3(c) - 2c_2^2 \end{cases} \end{aligned}$$

These equations immediately imply the relation (7). This proves that this condition is necessary for integrability of infinitesimal deformations.

5.4. *Integrability condition and the universal formula (5).* Let us prove that the condition (7) is sufficient for existence of a polynomial deformation. However, it is very difficult to solve the overdetermined system (10) directly. We will use the formula (5).

The formula (5) obviously defines a deformation (9) which is polynomial in  $\lambda, \mu$ . The first two coefficients  $c_0$  and  $c_1$  of this deformation are:

$$\begin{aligned} c_0 &= \lambda, \\ c_1 &= \frac{\lambda^2}{2} - \frac{\mu^2}{2}, \end{aligned} \tag{11}$$

and can be taken as *independent parameters*.

Fix the values of  $c_0$  and  $c_1$  and consider the condition (7) as a quadratic equation with  $c_2$  undetermined. The two solutions can be written (using (11)) as expressions from  $\lambda$  and  $\mu$ :

$$\begin{aligned} c_2^+ &= \frac{\lambda^3}{6} - \frac{\lambda\mu^2}{2} + \frac{\mu^3}{3}, \\ c_2^- &= \frac{\lambda^3}{6} - \frac{\lambda\mu^2}{2} - \frac{\mu^3}{3}. \end{aligned}$$

The expression  $c_2^+$  coincides with the third coefficient in the formula (5'). The deformation  $\pi^{\lambda, \mu}$  defined by the formula (5), corresponds to the infinitesimal deformation with  $c_0, c_1$  given by (11) and  $c_2 = c_2^+$ . The expression  $c_2^-$  can be obtained from  $c_2^+$  using the involution:  $*$  :  $(\lambda, \mu) \mapsto (\lambda, -\mu)$ .

We have shown that, every infinitesimal deformation satisfying (7) corresponds to a polynomial in  $c_0, c_1$  and  $c_2$  deformation. Indeed, it follows from existence of the formula (5) that the system (10) has a solution.

Theorem 5.1 is proven.  $\square$

*Remarks.* (a) The parameter  $\mu$  is a rational parameter on the semi-cubic parabola (7').  
Indeed,  $\tilde{c}_1 = \mu^2, \tilde{c}_2 = \mu^3$ .  
(b) Suppose that  $c_0, c_1, c_2 \in \mathbf{R}$ , then  $\lambda$  and  $\mu$  in (11) are real if and only if  $c_0^2 \geq 2c_1$ .

5.5. *Proof of Theorem 1.* We have shown that:

- (a) Every integrable infinitesimal deformation is equivalent to (6) and obeys the condition (7).
- (b) Every polynomial deformation is equivalent to a homogeneous one.
- (c) Given an infinitesimal deformation, there exists a unique homogeneous deformation corresponding to the infinitesimal one. It is given by the universal formula (5).

Theorem 1 is proven.  $\square$

## 6. Formal Deformations of the Embedding of $\text{Vect}(S^1)$ into $A(S^1)$

We classify formal deformations of the Lie subalgebra  $\text{Vect}(S^1)$  in  $A(S^1)$ .

**Theorem 2.** *Every formal deformation of the standard embedding  $\pi : \text{Vect}(S^1) \hookrightarrow A(S^1)$  is equivalent to one of the following deformations:*

$$\pi^t \left( f(x) \frac{d}{dx} \right) = f \left( x + \frac{(1-\lambda)t}{\xi} \right) \xi + \lambda t f' \left( x + \frac{(1-\lambda)t}{\xi} \right), \quad (5'')$$

where  $\lambda \in \mathbf{R}$ , the right-hand side is a (Taylor) series in  $t$ .

In other words, there exists a one-parameter family of formal deformations. The explicit formula for (5'') is:

$$\pi^t \left( f(x) \frac{d}{dx} \right) = \sum_{k=0}^{\infty} (1 - (k-1)\lambda)(1-\lambda)^{k-1} \frac{t^k}{k!} f^{(k)}(x) \xi^{-k+1}, \quad (5''')$$

note that  $\pi^t(f(x)d/dx) = f\xi + tf' + \dots$ .

Classification of infinitesimal deformation was done in Sect. 4.3. In order to prove Theorem 2, let us first classify the infinitesimal deformations which correspond to formal ones.

### 6.1. Integrable infinitesimal deformations.

**Proposition 6.1.** *An infinitesimal deformation (6) correspond to a formal deformation if and only if  $c_1 = c_2 = 0$ .*

A 1-cocycle on  $\text{Vect}(S^1)$  corresponding to an integrable infinitesimal deformation is, therefore, proportional to the cocycle  $C_0$  from Sect. 4.2. In other words, a nontrivial formal deformation is equivalent to a deformation of the form:

$$\pi^t \left( f(x) \frac{d}{dx} \right) = f\xi + tf' + (t^2)$$

(the constant  $c_0$  in (6) can be chosen:  $c_0 = 1$  up to normalization).

*Proof of the proposition.* Consider an infinitesimal formal deformation:

$$f(x) \frac{d}{dx} \mapsto f\xi + t(c_0 f' + c_1 f'' \xi^{-1} + c_2 f''' \xi^{-2})$$

of the standard embedding  $\pi : \text{Vect}(S^1) \rightarrow A(S^1)$ . One must show that it corresponds to a formal deformation  $\pi^t$  if and only if  $c_1 = c_2 = 0$ .

First, note that each term  $\pi_k$ ,  $k \geq 1$  of a formal deformation  $\pi^t$  (of the expansion (2)) can be chosen in a homogeneous form:

$$\pi_k \left( f(x) \frac{d}{dx} \right) = \sum_j \alpha_j^k f^{(j+1)} \xi^j, \tag{12}$$

where  $\alpha_j^k$  are some constants. The proof of this fact is analogous to the one of Lemma 5.2.

Second, apply the Nijenhuis–Richardson deformation relation (4) (which is equivalent to the homomorphism relation  $\pi([f, g]) = \{\pi(f), \pi(g)\}$ ). In the same way as in Sects. 5.2 and 5.3, collecting the terms with  $t^k$ , one obtains the following conditions:

(a) terms with  $t^2$ :

$$\begin{aligned} 2\alpha_3^2 &= 2c_0c_2 - c_1^2, \\ 5\alpha_4^2 &= -c_1c_2, \\ \begin{cases} \alpha_5^2 = 0 \\ 5\alpha_5^2 = -2c_2^2 \end{cases} &, \end{aligned}$$

and therefore,  $c_2 = 0$ .

(b) terms with  $t^3$ :

$$\begin{cases} 9\alpha_3^3 = -c_1\alpha_3^2 = (1/2)c_1^3 \\ 5\alpha_5^3 = 3c_1\alpha_3^2 = -(3/2)c_1^3 \end{cases} ,$$

and therefore,  $c_1 = 0$ .

Proposition 6.1 is proven.  $\square$

**Lemma 6.2.** *The constants  $\alpha_j^k$  in each term  $\pi_k$  (given by the formula (12)) of the deformation  $\pi^t$  satisfy the condition:  $\alpha_j^k = 0$  if  $j \geq k$ .*

*Proof.* First, one easily shows that  $\alpha_j^k = 0$  if  $j > k$ .

In the same way, using the identities (10), one obtains:  $\alpha_k^k = 0$  for every  $k \geq 1$ .

Lemma 6.2 is proven.  $\square$

For example, collecting the terms with  $t^4$  one has  $9\alpha_5^4 = 4c_0\alpha_4^3 - \alpha_1^2\alpha_3^2 = 0$  and  $\alpha_5^3 = 4\alpha_1^2\alpha_3^2 - 2(\alpha_2^2)^2 = -2(\alpha_2^2)^2$ , from where  $\alpha_2^2 = 0$ .

**Lemma 6.3.** *Every formal deformation  $\pi^t$  is equivalent to a formal deformation given by:*

$$\pi^t \left( f(x) \frac{d}{dx} \right) = f\xi + t f' + \alpha_1 t^2 f'' \xi^{-1} + \alpha_2 t^3 f''' \xi^{-2} + \sum_{k \geq 3} \alpha_k t^{k+1} f^{(k+1)} \xi^{-k},$$

where  $\alpha_i$  are some constants.

This means one can take in (12)  $\alpha_j^k = 0$  if  $j \leq k - 2$ .

*Proof.* Every formal deformation is equivalent to a deformation with  $\alpha_0^k = 0$  in (12). Indeed, constant  $\alpha_0^k$  is just the coefficient behind  $t^k f'$ . It can be removed (up to equivalence) by choosing a new formal parameter of deformation  $\tilde{t} = t + t^k \alpha_0^k$ .

Now, the lemma follows from Proposition 6.1 and homogeneity of the homomorphism condition. Indeed, the terms with  $j \leq k - 2$  are independent and therefore, the first nonzero term (corresponding to the minimal value of  $j$ ) must be a 1-cocycle. In the same way as in Lemma 5.2, one shows that such a 1-cocycle is trivial and can be removed up to equivalence.

Lemma 6.3 is proven.  $\square$

Now, the expressions  $P_k = \alpha_k t^{k+1}$  satisfy the identities (10). Thus, the deformation  $\pi^t$  is given by the formula (5) with  $\lambda = t$ .

Theorem 2 is proven.  $\square$

## 7. Some Properties of the Main Construction

Let us study some geometric and algebraic properties of the two-parameter deformation (5).

*7.1. Deformation of  $SL_2(\mathbf{R})$ -moment map.* Consider the standard Lie subalgebra  $sl_2(\mathbf{R}) \subset \text{Vect}(\mathbf{R})$  generated by the vector fields:

$$\frac{d}{dx}, x \frac{d}{dx}, x^2 \frac{d}{dx}.$$

For every  $\lambda$  and  $\mu$ , the restriction of the map  $\pi^{\lambda, \mu}$  given by the formula (5) to  $sl_2(\mathbf{R})$ , defines a Hamiltonian action of  $sl_2(\mathbf{R})$  on the half-plane  $\mathcal{H} = \{(x, \xi) \mid \xi > 0\}$  endowed with the standard symplectic structure:  $\omega = dx \wedge d\xi$ . Indeed, the formal series (5') in this case has only a finite number of nonzero terms and associates to each element of  $sl_2(\mathbf{R})$  a well-defined Hamiltonian function on  $\mathcal{H}$ .

Given a Hamiltonian action of a Lie algebra  $\mathfrak{g}$  on a symplectic manifold  $M$ , let us recall the notion of so-called *moment map* from  $M$  into the dual space  $\mathfrak{g}^*$  (see [5]). One associates to a point  $m \in M$  a linear function  $\tilde{m}$  on  $\mathfrak{g}$ . The definition is as follows: for every  $x \in \mathfrak{g}$ ,

$$\langle \tilde{m}, x \rangle := F_x(m),$$

where  $F_x$  is the Hamiltonian function corresponding to  $x$ . If the Hamiltonian action of  $\mathfrak{g}$  is homogeneous, then the image of the moment map is a coadjoint orbit of  $\mathfrak{g}$ .

In the case of  $sl_2(\mathbf{R})$ , the coadjoint orbits on  $sl_2(\mathbf{R})^* (\simeq \mathbf{R}^3)$  can be identified with level surfaces of the Killing form. Explicitly, for the coordinates on  $sl_2(\mathbf{R})^*$ , dual to the chosen generators of  $sl_2(\mathbf{R})$ :

$$y_1 y_3 - y_2^2 = \text{const.}$$

Thus, coadjoint orbits of  $sl_2(\mathbf{R})$  are cones (if the constant in the right hand side is zero), one sheet of a two-sheets hyperboloid (if the constant is positive), or a one-sheet hyperboloid (if the constant is negative).

**Proposition 7.1.** *The image of the half-plane ( $\xi > 0$ ) under the  $SL_2(\mathbf{R})$ -moment map is one of the following coadjoint orbits of  $sl_2(\mathbf{R})$ :*

(i)  $\lambda = 0$  or  $\mu = 0$ , the nilpotent conic orbit;

- (ii)  $\lambda\mu > 0$ , one sheet of a two-sheets hyperboloid;
- (iii)  $\lambda\mu < 0$ , a one-sheet hyperboloid.

*Proof.* The Poisson functions corresponding to the generators of  $sl_2(\mathbf{R})$  are:

$$\begin{aligned} F_1 &= \xi, \\ F_2 &= x\xi + \lambda, \\ F_3 &= x^2\xi + 2\lambda x + \lambda(\lambda - \mu)\xi^{-1}, \end{aligned}$$

respectively. These functions satisfy the relation:  $F_1 F_3 - F_2^2 = \lambda\mu$ . □

**7.2. The Virasoro algebra and central extension of the Lie algebra  $C^\infty(\mathbf{T}^2)$ .** Consider the Lie algebra  $C^\infty(\mathbf{T}^2)$  of smooth functions on the two-torus with the standard Poisson bracket. This Lie algebra has a two-dimensional space of nontrivial central extensions:  $H^2(C^\infty(\mathbf{T}^2)) = H^2(\mathbf{T}^2) = \mathbf{R}^2$ . The corresponding 2-cocycles were defined by A.A. Kirillov [4] (see also [11]):

$$c(F, G) = \int_\gamma F dG,$$

where  $F = F(x, y), G = G(x, y)$  are periodic functions:  $F(x + 2\pi, y) = F(x, y + 2\pi) = F(x, y)$  and  $\gamma$  is a closed path.

Recall that the *Virasoro algebra* is the unique (up to isomorphism) nontrivial central extension of  $\text{Vect}(S^1)$ . It is given by the so-called Gelfand-Fuks cocycle:

$$w(f(x)d/dx, g(x)d/dx) = \int_0^{2\pi} f'(x)g''(x) dx.$$

Let us show how the central extensions of  $C^\infty(\mathbf{T}^2)$  are related to the Virasoro algebra via the embedding (5).

Let  $\text{Vect}_{\text{Pol}}(S^1)$  be the Lie algebra over  $\mathbf{C}$  of polynomial vector fields on  $S^1$ . It is generated by:  $L_n = z^{n+1}d/dz$ , where  $z = e^{ix}$ . The formula (5) with  $\xi = e^{iy}$  defines a family of embeddings of  $\text{Vect}_{\text{Pol}}(S^1)$  into  $C^\infty(\mathbf{T}^2)_{\mathbf{C}}$ .

It is easy to show that the restriction of two basis Kirillov's cocycles to the subalgebra  $\text{Vect}_{\text{Pol}}(S^1) \hookrightarrow C^\infty(\mathbf{T}^2)_{\mathbf{C}}$  is proportional to the Gelfand-Fuks cocycle:

$$\left( \int_{\xi=\text{const}} F dG \right) \Big|_{\text{Vect}_{\text{Pol}}(S^1)} = \lambda^2 w \quad \text{and} \quad \left( \int_{x=\text{const}} F dG \right) \Big|_{\text{Vect}_{\text{Pol}}(S^1)} = \lambda^2 \mu^2 w.$$

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