SELF-ADJOINT DIFFERENTIAL OPERATORS AND CURVES ON A LAGRANGIAN GRASSMANNIAN, SUBJECT TO A TRAIN

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The systematic study of geometric curves determined by differential equations was initiated by Poincaré [1]. Presently, this theory has the most varied kinds of applications. Thus, the multidimensional generalization of the Sturm theory, proposed by Arnol'd [3], describes the properties of a curve on a Lagrangian Grassmannian, given by the evolution of a Lagrangian plane in symplectic space, under the action of a system of linear Hamiltonian equations.

In this paper, linear differential equations given by arbitrary scalar self-adjoint differential operators

$$L = (d/dt)^{2n} + \sum_{i=1}^{n} (d/dt)^{n-i} u_{n-i}(t) (d/dt)^{n-i} \tag{1}$$

with smooth real coefficients, are considered. Such equations reduce to Hamiltonian systems of a special form. Therefore, curves on a Lagrangian Grassmannian Λ_n, satisfying additional conditions, are also associated with them. At each point, the velocity of such a curve is tangent to the minimal stratum of a train with a vertex at the given point, and the acceleration vector is tangent to the second stratum etc. It is said that these curves are subject to a train.

The train of any point Λ_n is transversally oriented by the directions of the positive vectors (see [2-4]). It turns out that a curve, subject to a train, satisfies a universal property. In a little time, its point merges in a sign-fixed (positive or negative depending on the orientation) direction.

In [5] a multidimensional Lagrangian analogue was proposed of the Schwarz derivative, recovering the system of Newton's equations for a curve in Λ_n with positive velocity. It turns out that to each curve in Λ_n subject to a train, there corresponds a unique (with a precision up to sign) operator (1). The procedure of recovering the operator for a curve is the scalar version of the Lagrangian Schwarz derivative.

In the case when the coefficients of the operator (1) are periodic, the definition of the Maslov–Arnol'd index corresponding to a curve in Λ_n is given in the paper. This number is an invariant, relative to homotopies with a fixed monodromy operator. Its values can differ by an even number. For second-order equations, this invariant equals the mean number of rotations in a period, performed by an evolving line in phase space. Apparently, its existence was first noted by Poincaré (see also [6]).

Furthermore, an analogue of the Sturm theorem for equations given by the operators (1), will be presented. One of them – the nonvariability theorem – is contained in [7]. These theorems are consequences of the property of curves subject to a train. However, a simple deduction of them from the theorems of [2] also exists. I am obligated for this proof to V. I. Arnol'd and B. A. Khesin.

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1. Geometric Realization

1.1. Nonflattening Self-Dual Curves in Projective Space. To the equation $Ly = 0$ corresponds a curve in $(2n - 1)$-dimensional projective space; to each t corresponds a one-dimensional subspace in the space of solutions, consisting of solutions vanishing at t along with $(2n - 2)$ derivatives. Thereby, the curve $\ell(t)$ is given in $\mathbf{RP}^{2n-1}$, the projective closure of the solution space.

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Definitions. 1. By an accompanying flag of a curve f(t) in projective space $\mathbf{P}^k$ is meant a chain of subspaces $V_1(t) \subseteq V_2(t) \subseteq \ldots \subseteq V_k(t) \subseteq \mathbf{P}^k$, such that (in an arbitrary affine chart) each subspace $V_i(t)$ is spanned by the vectors f(t), f(t),...,$f^{(i)}(t)$. This definition does not depend on the choice of the parameter t on the curve and the affine chart on $\mathbf{P}^k$.

2. A curve $f(t) \subset \mathbf{P}^k$ is called nonflattening if at each point of it, the accompanying flat is complete: $V_1 \subset V_2 \subset \ldots \subset V_k = \mathbf{P}^k$.

3. A hyperplane $V_{k-1}(t) \subset \mathbf{P}^k$ is called osculating. A family of osculating hyperplanes gives a dual curve $f^*(t)$ in $\mathbf{P}^{k*}$ (see [4]). A nonflattening curve $f(t) \subset \mathbf{P}^k$ is self-dual if after identifying $\mathbf{P}^{k*}$ and $\mathbf{P}^k$, preserving the projective structure, the dual curve turns into a projectively equivalent one, given by $f^* = Af$, where $A \in \mathrm{PGL}(k+1)$.

LEMMA 1. There exists a one-to-one correspondence between the space of differential equations of the form Ly = 0 given by the operators (1) and the set of classes projectively equivalent to the self-dual curves in $\mathbf{RP}^{2n-1}$.

Proof. In the sapce of solutions of the equation Ly = 0, there exists a natural symplectic structure (the analogue of the Wronskian). Let us denote by L_i the differential operator $(d/dt)^{2n-i} + \sum_{i \leqslant k \leqslant n-1} (d/dt)^{k-i} u_k(t) (d/dt)^k$. Then

$$\omega(y, z) = \sum_{i=1}^{n} (-1)^i \begin{vmatrix} y^{(i-1)} & z^{(i-1)} \\ L_i y & L_i z \end{vmatrix}. \tag{2}$$

This expression does not depend on t if y, z are solutions of (1). On the projective closure of the solution space of $\cong \mathbf{RP}^{2n-1}$, there arises a contact structure (a contact hyperplane at the point $p \in \mathbf{RP}^{2n-1}$ is the projective closure of the hyperspace in $\mathbf{R}^{2n}$, skew-orthogonal to the line p).

Let us show that the curve $\ell(t)$ is self-dual. The contact structure on $\mathbf{RP}^{2n-1}$ naturally identifies it with $\mathbf{RP}^{2n-1*}$. Moreover, the curve dual to $\ell(t)$ turns into $\ell(t)$. This means that the hyperplane osculating with $\ell(t)$ in $\mathbf{RP}^{2n-1}$ coincides with the contact one. Actually, the hyperplane osculating with $\ell(t)$ at t_0 is the projective closure of the subspace consisting of solutions equal to zero at t_0, which by (2) is skew-orthogonal to the line $\ell(t_0)$.

In its turn, the self-dual curve in $\mathbf{RP}^{2n-1}$ uniquely determines the equation Ly = 0, where L is a self-adjoint operator. The lemma is proved.

1.2. Curves on a Lagrangian Grassmannian, Subject to a Train. Definition. By the train S_γ of a given point $\gamma \in \Lambda_n$, is meant the set of all Lagrangian subspaces nontransversal to γ, and the given point is a vertex of the train.

A train is a stratified submanifold with singularities in Λ_n; the stratum S_γ^k consists of the Lagrangian subspaces intersecting the vertex γ along a subspace of dimension n − k. The complement $S_\gamma^k \setminus S_\gamma^{k-1}$ is a smooth submanifold. For these deformations and properties, see [2-4].

Locally Λ_n is identified with the space of quadratic forms in $\mathbf{R}^n$, and the train of the given point — with the set of degenerate forms. Let us fix an arbitrary polarization — a pair of transversal Lagrangian subspaces α, β in $(\mathbf{R}^{2n}, \omega)$. By a joint form $\Phi_{[\alpha,\beta]}$ is meant a quadratic form in $(\mathbf{R}^{2n}, \omega)$, $\Phi_{[\alpha,\beta]}(\zeta) = \omega(\xi, \eta)$, where ξ is the projection of ζ on α along β, and η — on β along α. To each Lagrangian subspace λ, let us associate the restriction $\Phi_{[\alpha,\beta]}|_\lambda$, called a generating function. It is clear that if α = λ, then $\Phi_{[\alpha,\beta]}|_\lambda = 0$, and the set of degenerate forms corresponds to the train λ.

Definition. Let us fix the polarization (α, β). Then for any curve $\gamma(t) \subset \Lambda_n$, along with the velocity vector $\dot\gamma(t)$, the vectors $\ddot\gamma(t)$, $\gamma^{(3)}(t)$,... are also defined. Let us call a curve γ(t) subject to a train if each of the vectors $\gamma^{(k)}(t)$ for k < n is tangent to $S^k_{\gamma(t)}$ and transversal to $S^{k-1}_{\gamma(t)}$, and the vector $\gamma(t)^{(n)}$ is transversal to $S^{n-1}_{\gamma(t)} = S_{\gamma(t)}$. It is clear that this definition does not depend on the choice of polarization.

To the linear differential equation Ly = 0 given by a scalar self-adjoint operator L of operator 2n, corresponds the curve λ(t) in Λ_n. To each t corresponds an n-dimensional subspace in the solution space with null boundary conditions at the point t: $\lambda(t) = \{y \mid y(t) = \ldots = y^{(n-1)}(t) = 0\}$. The Lagrange nature of the plane λ(t) follows immediately from (2).

Assertion. The curve λ(t) is subject to a train.

Proof. The vector $\lambda^{(k)}(t)$ is directed along the submanifold Λ_n consisting of Lagrangian subspaces intersecting $\lambda(t)$ along the $(n - k)$-dimensional subspace $\lambda^{(k)}(t) = \{y \mid y(t) = \ldots = y^{(n-k-1)}(t) = 0\}$.

THEOREM. There exists a one-to-one correspondence between the linear differential equations of the form Ly = 0, where L is the self-adjoint differential oprator (1), and the classes Sp(2n) are equivalent curves in Λ_n subject to a train.

Proof. Let $\lambda(t)$ be a curve in Λ_n, subject to a train. In each Lagrangian subspace $\lambda(t) \subset (\mathbf{R}^{2n}, \omega)$ there exists a "most slow" line, determined in the following manner. Let us fix the polarization (α, β) (giving a chart on Λ_n). The vectors $\dot{\lambda}, \ddot{\lambda}, \ldots, \lambda^{(n-1)}$ are tangent to the submanifold in Λ_n consisting of Lagrangian subspaces intersecting $\lambda(t_0)$ along some one-dimensional subspace $V(t_0)$. Let us show that the curve V(t) in $\mathbf{RP}^{2n-1}$ is self-dual. For this, it is sufficient that for any t, the hyperplane osculating with V(t) in $\mathbf{RP}^{2n-1}$ coincide with the contact one (see the proof of Lemma 1).

Let us, in an arbitrary manner, lift the curve $V(t) \subset \mathbf{RP}^{2n-1}$ to the curve $\tilde{V}(t)$ in the symplectic space $(\mathbf{R}^{2n}, \omega)$. The accompanying flag of the curve $V(t)$: $V^s(t) \subseteq \ldots \subseteq V^{2n-1}(t)$ is lifted to a chain of linear subspaces $\tilde{V}_i(t)$ in $(\mathbf{R}^{2n}, \omega)$. It is necessary to show that $\tilde{V}(t)^{\perp} = \tilde{V}^{2n-1}(t)$. The flag $V(t) = \langle \tilde{V}(t) \rangle \subseteq \tilde{V}^1(t) \subseteq \ldots \subseteq \tilde{V}^{2n-1}(t) \subset \mathbf{R}^{2n}$ is called the accompanying flag of the curve $\tilde{V}(t)$.

LEMMA 2. The accompanying flag of the curve $\tilde{V}(t) \subset (\mathbf{R}^{2n}, \omega)$ projecting to the curve $V(t) \subset \mathbf{RP}^{2n-1}$, satisfying the conditions: a) $\tilde{V}^{n-1}(t)$ coincides with the Lagrangian subspace $\lambda(t) \subset (\mathbf{R}^{2n}, \omega)$; b) $\tilde{V}^{n+k} = \tilde{V}^{n-k-2\perp}$.

Proof. a) Let us denote by F(t) the symmetric operator from the Lagrangian subspace $\lambda(t)$ to the dual, giving the quadratic form $\Phi_{[\alpha,\beta]}|_{\lambda}$ (generating function).

The subjection to the train of the curve $\lambda(t) \subset \Lambda_n$, means that $\lambda \supset \operatorname{Ker} \dot{F} \supset \operatorname{Ker} \ddot{F} \supset \ldots \supset \operatorname{Ker} F^{(n-1)} \simeq \mathbf{R}$ is a complete flag. It follows from the definition that $V(t) = \bigcap_{1 \leqslant i \leqslant n-1} \operatorname{Ker} F^{(i)} = \operatorname{Ker} F^{(n-1)}$.

Let us identify the plane $\lambda(t)$ with α, having associated to each vector $Y \in \lambda(t)$, its projection on α along βY_{α}, and the effective space $\lambda'(t)$ — with β; let $Y \in \lambda(t), Z \in \beta$ let $\langle Y, Z \rangle = \omega(Y, Z) = \omega(Y_{\alpha}, Z)$. After this, the symmetric operator $F: \lambda \to \lambda'$ is realized as an operator from α to β, such that for any vector $Y \in \lambda(t)$ $FY_{\alpha} = Y_{\beta}$. Since the generating function determines the plane $\lambda(t)$, this equation is cahracteristic for the vectors $\lambda(t)$.

From the fact that the vector $\tilde{V}(t)_{\alpha} \in \bigcap_{1 \leqslant i \leqslant n-1} \operatorname{Ker} F^{(i)}(t)$, it immediately follows that $\tilde{V}_{\alpha}^{(k)} = (d^k\tilde{V}(t)/dt^k)_{\alpha} \in \operatorname{Ker} F^{(n-k-1)}(t)$ for $k \leqslant n-2$. For example, $F^{(n-2)}V \equiv 0$ entails $(d/dt)(F^{(n-2)}\tilde{V}_{\alpha}) = F^{(n-1)}\tilde{V}_{\alpha} + F^{(n-2)}\dot{\tilde{V}}_{\alpha} = 0$, whence $\dot{\tilde{V}}_{\alpha} \in \operatorname{Ker} F^{(n-2)}$.

Let us show, finally, that for $k \leqslant n-1$ $\tilde{V}_{\beta}^{(k)} = F\tilde{V}_{\alpha}^{(k)}$, whence it will follow that the subspace $\tilde{V}^k \subseteq \lambda(t)$. Actually, let $\tilde{V}^{(k-1)} \in \lambda(t)$, that is $\tilde{V}_{\beta}^{(k-1)} = F\tilde{V}_{\alpha}^{(k-1)}$. Then $\tilde{V}_{\beta}^{(k)} = (d/dt) F\tilde{V}_{\alpha}^{(k-1)} = \dot{F}\tilde{V}_{\alpha}^{(k-1)} + F\tilde{V}_{\alpha}^{(k)} = F\tilde{V}_{\alpha}^{(k)}$.

The vectors $\tilde{V}, \dot{\tilde{V}}, \ldots, \tilde{V}^{(n-1)}$ are linearly independent, since $\tilde{V}^{(n)} \notin \lambda(t)$ [this immediately follows from the definition of the curve V(t)]. Therefore, the subspace spanned by them is $\lambda(t)$.

b) The velocity vector $\dot{\lambda}$ is tangent to the manifold of Lagrangian subspaces intersecting λ along the subspace $\tilde{V}^{n-2}$ (since $\tilde{V}_{\alpha}^{(n-2)} \in \operatorname{Ker} \dot{F}$). The collection of all such Lagrangian subspaces is an $(n+1)$-dimensional subspace in $(\mathbf{R}^{2n}, \omega)$ $V^{n-2\perp}$. Since the vector $\tilde{V}^{(n-1)}(t) \in \lambda(t)$, its velocity $\tilde{V}^{(n)}$ is tangent to the subspace $\tilde{V}^{n-2\perp}$, and this means (by linearity) $\tilde{V}^{(n)} \in \tilde{V}^{n-2\perp}$. Similarly $\tilde{V}^{(n+k)} \in \tilde{V}^{n-k-2\perp}$. The lemma is proved.

In this manner, the curve V(t) in $\mathbf{RP}^{2n-1}$ is self-dual. By Lemma 1, a certain equation Ly = 0 corresponds to it. Therefore, along each curve subject to a train, the equation is uniquely recovered. Moreover, the same equation corresponds to the Sp(2n)-equivalent curve in Λ_n. The theorem is proved.

Remark 1. Let us lift the curve V(t) to the curve $\tilde{V}(t)$ in $(\mathbf{R}^{2n}, \omega)$, so that the volume of the parallelepiped spanned by the vectors $\tilde{V}(t), \dot{\tilde{V}}(t), \ldots, \tilde{V}^{(2n)}(t)$ (relative to the form of the volume $\Lambda^n\omega$) is constant. Knowing the velocity of each of the vectors $\tilde{V}, \ldots, \tilde{V}^{(2n)}$ in $(\mathbf{R}^{2n}, \omega)$, one can determine a linear vector field on $(\mathbf{R}^{2n}, \omega)$. Lemma 2 is equivalent to its Hamiltonicity.

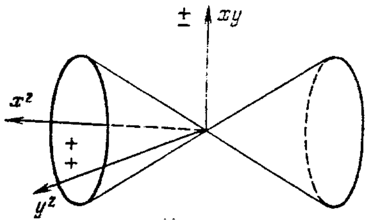

Fig. 1

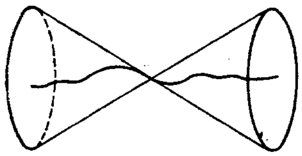

Fig. 2

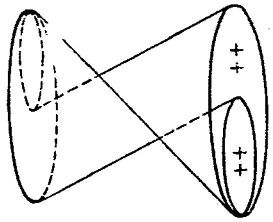

Fig. 3

Remark 2. An explicit formula for the generating function F(t) of the plane $\lambda(t)$ can prove to be useful. Let $y_1(t),\ldots,y_n(t)$, $y_1'(t),\ldots,y_n'(t)$ be a canonical basis in the symplectic space of solutions of the equations Ly = 0, where $\{y_i\}$ is a basis in the plane α and $\{y_j'\}$ — in β. Let us denote by Y the matrix $(y_i^{(j)}(t))$, and by Y' — the matrix $(y_i'^{(j)}(t))$. Then in the polarization (α, β) in the basis in the plane $\lambda(t)$, projecting to $y_1',\ldots,y_n'$ (along α), the generating function is given by the symmetric matrix $F = (YY'^{-1}/2)$.

2. Theorem of the Intersection with a Train

The train of any point of a Lagrangian Grassmannian is transversally oriented to the directions of the positive vectors (velocities of motion of the Lagrangian planes under the action of positive-definite Hamiltonians) (see [2]). At each point, the positive vectors form an open convex cone Eq. (1). In the language of generating functions, the cone of positive vectors at the point is identified with the cone of positive-definite quadratic forms.

THEOREM (on the intersection with a train). A curve $\lambda(t)$ in Λ_n, subject to a train, merges in a small time in a sign-fixed (positive or negative) direction with respect to a train with a vertex at the given point (Fig. 2).

Actually, the cause of this property is the mutual disposition of two trains with vertices in nontransversal planes (Fig. 3). There is the geometric version of the Rayleigh–Gårding inequality [3]: if the plane λ_2 belongs to the positive part of some stratum of the train of the point λ_1, then the cone of positive vectors with a vertex at λ_2 lies wholly in the positive cone with vertex at λ_1.

Proof. According to the theorem proved, the curve is always given by the equation Ly = 0. The simple proof presented below was suggested to the author by V. I. Arnol'd and B. A. Khesin.

LEMMA 3. The equation Ly = 0 given by operator (1), determines a Hamiltonian vector field

$$\begin{aligned} \dot{q}_i &= q_{i+1} && (1 \leqslant i \leqslant n-1), \\ \dot{q}_n &= p_n, && \\ \dot{p}_i &= p_{i-1} - u_{i-1}(t)\, q_i && (1 \leqslant i \leqslant n) \end{aligned} \tag{3}$$

on a symplectic phase space with the symplectic form $\omega = \sum (-1)^{i+1}\, dp_i \wedge dq_i$. Its Hamiltonian $H = (-1)^{n+1} p_n^2 + \sum_{i=1}^{n-1} (-1)^{i+1} [p_i q_{i+1} + u_{i-1} q_i^2]$.

Proof of the Lemma. The space of solutions of the equation Ly = 0 is identified with the space:

$$q_i = y^{(i-1)}(0);\ p_i = L_i y(0).$$

Moreover, the form (2) turns into ω, and the equation into the vector field (3).

The restriction for the Hamiltonian H on the vertical plane equals $H|_{q=0} = (-1)^{n+1} p_n^2$, therefore, H is the limit for the family of the Hamiltonians, sign-fixed on the vertical plane:

$$H_\varepsilon = H + (-1)^{n+1} \varepsilon \sum p_i^2.$$

For simplicity, we will further conduct the argument, in the positive case (n is odd). The index of the intersection of any Lagrangian plane, evolving under the action of the Hamiltonian vector field with the Hamiltonian H_ε, with the train of a vertical plane, is positive, the velocity of the evolution of the same plane (q = 0) is also positive for any t [3]. Therefore, there exists a small interval of time $(t, t + \delta)$ not depending on ε, in which the evolution of the plane (q = 0) occurs in a positive component, relative to a proper train. In this manner, the evolution under the action of the Hamiltonian H must occur in a nonnegative component and the plane $\lambda(t + \delta)$ in the solution space is found in a nonnegative component in relation to $\lambda(t)$. Now the assertion of the theorem follows from the transversality of $\lambda(t)$ and $\lambda(t + \delta)$ for small δ (the uniqueness theorem).

3. Sturm Theorems

Let α be an arbitrary Lagrangian subspace in the solution space of the equations $Ly = 0$. By the moment of verticality α is meant the value t, for which the curve $\lambda(t)$, corresponding to the intersection, intersects the train S_α. This means that the plane α contains a solution equal to zero at the point t, along with the first $n - 1$ derivatives. The following Sturm theorem are proved by the limiting transition (see above) from the theorem of [3].

THEOREM (on zeros). The difference between the numbers of moments of verticality of two Lagrangian subspaces α, β on any segment does not exceed the number of changes of n.

THEOREM (on nonvariability) (see [7]). If the coefficients of the operator (1) satisfy the condition $(-1)^{n+i+1} u_i(t) \leqslant 0$, then the number of momens of verticality of any Lagrangian subspace in the solution space of Eq. (1) does not exceed n.

THEOREM (comparisons). If the coefficients of the two equations are connected by the inequality $(-1)^{n+i+1}(u_i^1(t) - u_i^2(t)) \leqslant 0$, then the number of moments of verticality of an arbitrary Lagrangian subspace in the space of solutions of the first equation, can exceed the number of moments of verticality of a Lagrangian subspace in the solution space of the second equation by no more than n.

4. Homotopic Poincaré Invariant

Let us consider the operator L with periodic coefficients. It follows from the theorem on zeros that the mean number of moments of verticality on the period is defined for an arbitrary Lagrangian plane in $(\mathbf{R}^{2n}, \omega)$, not depending on its choice. Indeed, in this case, the monodromy operator M (a shift by a period) is defined in the space of solutions, being a symplectic linear operator. Let T be the period. From the fact that for any k, the number of moments of verticality of the Lagrangian planes α and $M^k\alpha$ on an interval of length NT differ by no more than n, it follows that the limit of the ratio of the number of moments of verticality α to the number N exists as $N \to \infty$, not depending on the location of the interval. The independence of this number from the choice of the plane α, clearly, follows from the theorem on zeros. Let us denote it P(L).

THEOREM. The number P(L) is invariant with respect to homotopies with constant monodromy in the space of operators (1) with periodic coefficients. Its values can differ by an even number.

Proof. By the theorem on zeros, the mean number of moments of verticality of the Lagrangian plane coincides in the period with the time-averaged Maslov–Arnol'd index of a curve $\lambda(t)$ in Λ_n, independent of the homotopies of a curve with fixed ends. Since the end of the curve $\lambda(t)$ for $t \in [0, NT]$ is determined by its origin and the monodromy operator: $\lambda(t + NT) = M^N\lambda(t)$, then this is a number invariant with respect to homotopies with constant monodromy.

For two equations with identical monodromy operators, let the invariant under consideration assume different values. Then, there exists a time interval $(t_0, t_0 + T)$ on which the Maslov–Arnol'd indices of the corresponding curves $\lambda(t)$ and $\lambda'(t)$ in Λ_n are different. Let us identify the symplectic solution spaces so that the planes $\lambda(t_0)$ and $\lambda'(t_0)$, and also $\lambda(t_0 + T) = M\lambda(t_0)$ and $\lambda'(t_0 + T) = M\lambda'(t_0)$ coincide. Let us lift the curves $\lambda(t)$ and $\lambda'(t)$ in $\tilde{\Lambda}_n$, the universal covering of Λ_n, to the curves $\tilde{\lambda}(t)$ and $\tilde{\lambda}'(t)$, so that $\tilde{\lambda}(t_0) = \tilde{\lambda}'(t_0)$. The points $\tilde{\lambda}(t_0 + T)$ and $\tilde{\lambda}'(t_0 + T)$ differ by the action of an element of the group $\pi_2(\Lambda_n^+)$, where Λ_n^+ is the manifold of all oriented Lagrangian subspaces $(\mathbf{R}^{2n}, \omega)$ (a two-sheeted covering of Λ_n). Therefore, $m(\tilde{\lambda}(t_0 + T), \tilde{\lambda}'(t_0 + T))$ is an even number.

The Maslov–Arnol'd index of a closed loop on Λ_n equals the index of the intersection of it with the train of each point of Λ_n (see [3]). Therefore, the number of moments of verticality of any Lagrangian plane relative to the curves $\lambda(t)$ and $\lambda'(t)$ for $t \in (t_0, t_0 + T)$, differ by an even number, not depending on it. In particular, this is true for the planes $\alpha, M\alpha, M^2\alpha, \ldots$, whence the second assertion of the theorem follows.

In conclusion, let us formulate an interesting question: is the invariant P(L), the unique invariant of the self-adjoint operator (1) with periodic coefficients relative to the homotopies preserving its monodromy operator?

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ESTIMATE OF THE DERIVATIVE OF AN ALGEBRAIC POLYNOMIAL

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We denote by $\lambda(n)$, $n \in \mathbf{N}$ the smallest number for which the inequality

$$\iint_E |P'(z)|\,dx\,dy \leqslant \lambda(n)\cdot\sup_{z\in E}|P(z)|$$

is satisfied for an arbitrary algebraic polynomial of degree at most n and an arbitrary rectifiable set E belonging to the disk $|z| \leqslant 1$. It is easily seen from geometric considerations (see [1, 2]) that

$$\lambda(n) \leqslant \pi\sqrt{n}. \tag{1}$$

According to an hypothesis of Littlewood [1], as yet neither proved nor disproved, the relationship $\lambda(n) = O(n^{\theta})$ holds, where $\theta < 1/2$. Other equivalent formulations of this hypothesis are given in [1] (see also [2]). Recently Eremenko and Sodin [3] improved the obvious upper estimate (1) as follows: $\lambda(n) = o(\sqrt{n})$. In the present paper we obtain a new lower estimate for $\lambda(n)$.

THEOREM. Absolute positive constants A and a exist such that for arbitrary $n \geqslant 9$ the following inequality is satisfied:

$$\lambda(n) \geqslant A\exp(a\ln n/\ln\ln n).$$

We remark that up to the present time the best lower estimate for $\lambda(n)$ was due to Hayman [2]: $\lambda(n) \geqslant A\ln n$, where $n \geqslant 2$, $A > 0$ is an absolute constant. Interest in Littlewood's hypothesis may be explained by its applications in the theory of the distribution of the values of entire functions (see [1, 3]). The proof of our theorem is based on Lemmas 1-3, obtained below, and a theorem of Walsh–Russell [4].

LEMMA 1. Let f be a polynomial, $r > \rho > 0$, $\delta > 0$, $z_0 \in \mathbf{C}$ and $\varphi(z) = (z - z_0)/r$. If

$$|\varphi(z) - f(z)| \leqslant \delta \quad \text{for } |\varphi(z)| \leqslant 1,$$

then for $|\varphi(z)| \leqslant \rho/r$ we have the inequalities

$$\frac{1}{r} - \frac{r\delta}{(r-\rho)^2} \leqslant |f'(z)| \leqslant \frac{1}{r} + \frac{r\delta}{(r-\rho)^2}. \tag{2}$$

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