Note

Real Clifford Algebras and Quadratic Forms over $\mathbb{F}_2$: Two Old Problems Become One

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Importance of establishing connections between different branches of scientific enquiry; such steps are of the nature of revolutions in science.

W. K. Clifford

Clifford algebras are important for many areas of pure and applied mathematics and physics. Real Clifford algebras were introduced by the English mathematician and philosopher William Kingdon Clifford in 1878 [4]. Clifford called them "geometric algebras." They generalize complex numbers and quaternions, as well as multidimensional Grassmann algebras. Real Clifford algebras and their representations appeared independently in the work of Hurwitz on square identities [8], and still play an essential role in this subject. The application of real Clifford algebras in topology was initiated in [2].

Every real Clifford algebra is isomorphic to one of the algebras of real, complex, or quaternionic matrices whose size is a power of 2, or their "double copies," that is, the direct sum of one of these matrix algebras with itself. More precisely, one has the following theorem of Chevalley [3].

(1) There are exactly two real Clifford algebras of dimension 2^n for $n = 2k$, namely $\mathrm{Mat}(2^k, \mathbb{R})$ and $\mathrm{Mat}(2^{k-1}, \mathbb{H})$.
(2) There are exactly three real Clifford algebras of dimension 2^n for $n = 2k+1$: the algebra $\mathrm{Mat}(2^k, \mathbb{C})$, which is the only simple algebra in this case, and two double copies of the aforementioned algebras of real and quaternionic matrices.

The usual modern way to obtain this classification uses periodicity results, among which the Bott periodicity is the most famous; the reader is invited to consult the book [10], and an online course [7].

The problem of classification of quadratic forms over the field of two elements $\mathbb{F}_2$ was solved by Dickson [6]. Recall that the vector space $\mathbb{F}_2^n$ consists in n-tuples $x = (x_1, \ldots, x_n)$, where $x_i \in \{0, 1\}$. The classification result can be formulated as follows:

(1') There are exactly two equivalence classes of nondegenerate[1] quadratic forms on $\mathbb{F}_2^{2k}$; these classes can be represented by the forms:

$$\begin{aligned} \mathbf{q}_0(x) &= x_1x_2 + x_3x_4 + \cdots + x_{2k-1}x_{2k}, \\ \mathbf{q}_1(x) &= x_1x_2 + x_3x_4 + \cdots + x_{2k-1}x_{2k} + x_1 + x_2, \end{aligned}$$

where the summation is performed modulo 2. Note that the form $\mathbf{q}_1$ can also be written as a homogeneous quadratic form, since $x_i^2 = x_i$.

(2') The form

$$\mathbf{q}_2(x) = x_1x_2 + x_3x_4 + \cdots + x_{2k-1}x_{2k} + x_{2k+1},$$

is the only regular quadratic form on $\mathbb{F}_2^{2k+1}$. There are exactly three equivalence classes of quadratic forms of rank $2k$ on $\mathbb{F}_2^{2k+1}$, those of $\mathbf{q}_0, \mathbf{q}_1$, trivially extended to $\mathbb{F}_2^{2k+1}$, and $\mathbf{q}_2$.

Nonequivalence of the forms $\mathbf{q}_0$ and $\mathbf{q}_1$ follows from the *Arf invariant*, defined as follows: $\mathrm{Arf}(\mathbf{q}) = 1$ if the number of points $x \in \mathbb{F}_2^n$ such that $\mathbf{q}(x) = 1$ is greater than the number of points at which $\mathbf{q}(x) = 0$; otherwise, $\mathrm{Arf}(\mathbf{q}) = 0$. An easy computation then shows that

$$\mathrm{Arf}(\mathbf{q}_0) = 0, \qquad \mathrm{Arf}(\mathbf{q}_1) = 1.$$

The proof of the previous theorem is very simple (see Appendix 1).

The goal of this note is to explain that the noticeable similarity of the aforementioned classification theorems is not a coincidence: the problems are, indeed, equivalent. This equivalence is quite surprising, and can even be misunderstood. Every Clifford algebra is associated with a quadratic form, and every quadratic form defines a Clifford algebra (the algebras and the forms are over the same ground field). This is a classical and tautological relation, which is not an equivalence, as we will see. Here we compare Clifford algebras over $\mathbb{R}$ and quadratic forms over $\mathbb{F}_2$.

Quadratic forms over $\mathbb{F}_2$ are also very useful in topology, see, for example, [9]. Can this be a reason or a consequence for equivalency of two theories? In particular, it is amusing to know that the difference between real and quaternionic

[1] The terminology will be clarified in the section after the next.

 Volume 38, Number 3, 2016
DOI 10.1007/s00283-016-9637-2

matrices is measured by the Arf invariant. This could perhaps provide Clifford with some material as philosopher...

Definition of Real Clifford Algebras

The simplest definition of a real Clifford algebra is the original definition of Clifford [4].

The real Clifford algebra $\mathbf{Cl}_{p,q}$ is the associative algebra with unit $\mathbf{1}$ and $n = p + q$ generators $\iota_1, \ldots, \iota_n$ that anticommute:

$$\iota_i \iota_j = -\iota_j \iota_i, \quad i \neq j, \tag{1}$$

and square to $\mathbf{1}$ or $-\mathbf{1}$:

$$\iota_i^2 = \begin{cases} \mathbf{1}, & 1 \leq i \leq p \\ -\mathbf{1}, & p < i \leq n. \end{cases} \tag{2}$$

The pair of numbers (p, q) is called the signature. The monomials

$$\iota_I = \iota_{i_1} \cdots \iota_{i_k},$$

where $1 \leq i_1 < \ldots < i_k \leq n$, and $I = \{i_1, \ldots, i_k\}$ form a basis of $\mathbf{Cl}_{p,q}$ so that, $\dim \mathbf{Cl}_{p,q} = 2^n$.

A more general definition of Clifford algebras is as follows. Given a vector space V over a field $\mathbb{F}$, and a quadratic form $Q : V \to \mathbb{F}$, the corresponding Clifford algebra is the quotient of the tensor algebra $T(V)$ by the bilateral ideal generated by the elements of the form

$$v \otimes v - Q(v)\mathbf{1},$$

for all $v \in V$, and where $\mathbf{1}$ is the unit.

Choose a basis $\{\iota_1, \ldots, \iota_n\}$ of V, so that the Clifford algebra is generated by $\iota_1, \ldots, \iota_n$, with relations

$$\iota_i^2 = Q(\iota_i)\mathbf{1}, \qquad \iota_i \iota_j + \iota_j \iota_i = B(\iota_i, \iota_j)\mathbf{1},$$

where B is the *polar bilinear form* defined by

$$B(u, v) := Q(u + v) - Q(u) - Q(v).$$

We consider only the case where $\mathbb{F} = \mathbb{R}$ and the bilinear form B is nondegenerate. Choosing a basis in which $B(\iota_i, \iota_j) = \pm\delta_{i,j}$, one then goes back to the earlier mentioned Clifford definition. Note that, although real Clifford algebras are parametrized by two numbers (p, q), that is, the signature of the quadratic form, many of them are isomorphic. Therefore, there is no equivalence between real quadratic forms and real Clifford algebras.

Quadratic Forms over $\mathbb{F}_2$

A quadratic form $\mathbf{q} : \mathbb{F}_2^n \to \mathbb{F}_2$ can be written in coordinates as follows:

$$\mathbf{q}(x) = \sum_{1 \leq i \leq j \leq n} q_{ij}\, x_i x_j,$$

where the coefficients $q_{ij} \in \{0, 1\}$, and where the summation is modulo 2. Note that the "diagonal terms" of $\mathbf{q}$:

$$\sum_{1 \leq i \leq n} q_{ii}\, x_i^2$$

constitute its *linear part*, since $x_i^2 = x_i$.

A quadratic form on $\mathbb{F}_2^n$ also defines the polar bilinear form[2]

$$\mathbf{b}(x, y) := \mathbf{q}(x + y) + \mathbf{q}(x) + \mathbf{q}(y),$$

which is *alternating*, that is, $\mathbf{b}(x, x) = \mathbf{q}(0) = 0$. This implies that its rank is always even.

The relation between quadratic forms and the corresponding polar bilinear alternating forms is that the polar form $\mathbf{b}$ "forgets" about the linear part of $\mathbf{q}$. Two quadratic forms $\mathbf{q}$ and $\mathbf{q}'$ on $\mathbb{F}_2^n$ correspond to the same polar form if and only if $\mathbf{q} + \mathbf{q}'$ is a linear function. This is the main difference with the real case where the quadratic form can be reconstructed from the corresponding polar bilinear form.

The *rank* of a quadratic form on $\mathbb{F}_2^n$ is defined as the rank of the corresponding polar form. Therefore, a quadratic form over $\mathbb{F}_2$ can be *nondegenerate* (i.e., of full rank) only if $n = 2k$. If $n = 2k + 1$, and a quadratic form has rank $n - 1$, then it makes sense to ask if it is *regular*, that is, not equivalent to a form written in fewer than n variables.

Equivalence

We will now explain the equivalence of the theories of real Clifford algebras and quadratic forms on $\mathbb{F}_2^n$. The following crucial idea was suggested by Albuquerque and Majid [1].

Consider the vector space $\mathbb{R}[\mathbb{F}_2^n] \simeq \mathbb{R}^{2^n}$ with natural basis e_x, where $x \in \mathbb{F}_2^n$. Since $\mathbb{F}_2^n$ is an abelian group, the space $\mathbb{R}[\mathbb{F}_2^n]$ has the structure of a commutative algebra defined by

$$e_x e_y = e_{x+y}.$$

Identify the basis of $\mathbf{Cl}_{p,q}$ with the basis of $\mathbb{R}[\mathbb{F}_2^n]$ by

$$\iota_I \longleftrightarrow e_{x_I}, \tag{3}$$

where $x_I = (x_1, \ldots, x_n)$, such that $x_i = 1$, iff $i \in I$. The algebra $\mathbf{Cl}_{p,q}$ is thus identified with $\mathbb{R}[\mathbb{F}_2^n]$, as a vector space. However, the product in $\mathbf{Cl}_{p,q}$ and in $\mathbb{R}[\mathbb{F}_2^n]$ differs by a sign.

Consider a new product in $\mathbb{R}[\mathbb{F}_2^n]$:

$$e_x \cdot e_y := (-1)^{f(x,y)} e_{x+y},$$

where $f : \mathbb{F}_2^n \times \mathbb{F}_2^n \to \mathbb{F}_2$ is a function of two arguments. This new structure is an algebra called a *twisted group algebra*; we denote it $(\mathbb{R}[\mathbb{F}_2^n], f)$.

Proposition 0.1. [1]. *The algebra* $\mathrm{Cl}_{p,q}$ *is isomorphic to* $(\mathbb{R}[\mathbb{F}_2^n], f)$ *where f is the bilinear form*

$$f(x, y) = \sum_{1 \leq j < i \leq n} x_i y_j + \sum_{p+1 \leq i \leq n} x_i y_i.$$

Proof. Let us first check that the generators $e_{\mathbf{x}_i}$, where

$$\mathbf{x}_i = (0 \ldots 0\, 1\, 0 \ldots 0)$$

with 1 at the ith position, satisfy the relations (1) and (2). Indeed, $f(\mathbf{x}_i, \mathbf{x}_j) = 0$ and $f(\mathbf{x}_j, \mathbf{x}_i) = 1$, provided $i < j$, so that the generators anticommute. Furthermore, since $f(\mathbf{x}_i, \mathbf{x}_i) = 1$ for $i > p$, the generators $e_{\mathbf{x}_i}$ square to $-\mathbf{1}$ for $i > p$ and to $\mathbf{1}$ for $i \leq p$.

[2]Note that in characteristic 2, there is no difference between the "+" and "−" signs.

The algebra $(\mathbb{R}[\mathbb{F}_2^n], f)$ is associative, as readily follows from the fact that f is bilinear. Therefore, the map (3) is a homomorphism of $(\mathbb{R}[\mathbb{F}_2^n], f)$ to $\mathrm{Cl}_{p,q}$ with trivial kernel. □

The previous realization of real Clifford algebras as twisted group algebras was used in [1] to recover structural results, such as periodicities.

Let us go one step further and address the question of isomorphism of twisted group algebras with bilinear twisting functions. Define the "diagonal" quadratic form

$$\alpha(x) := f(x, x).$$

It turns out that the algebra is completely determined by α.

Theorem 1. *Two twisted group algebras $(\mathbb{R}[\mathbb{F}_2^n], f_1)$ and $(\mathbb{R}[\mathbb{F}_2^n], f_2)$ with bilinear functions f_1 and f_2, are isomorphic if and only if the corresponding quadratic forms, α_1 and α_2, are equivalent.*

Our proof consists of two parts. We first prove the "if" part, whereas the converse statement, based on classification of quadratic forms, will be proved in the end of the next section.

Consider a twisted group algebra $(\mathbb{R}[\mathbb{F}_2^n], f)$ with bilinear f, and two generators, e_i, e_j. Their commutation relation is determined by the value of $f(\mathbf{x}_i, \mathbf{x}_j) + f(\mathbf{x}_j, \mathbf{x}_i)$. It turns out that the symmetrization of f coincides with the polarization of α, that is,

$$f(x, y) + f(y, x) = \alpha(x + y) + \alpha(x) + \alpha(y),$$

for all $x, y \in \mathbb{F}_2^n$. Indeed, it suffices to check this for every monomial $x_i y_j$. Therefore, the quadratic form α completely determines the relations between the generators.

We proved that if $\alpha_1 = \alpha_2$, then the algebras are isomorphic. Since a twisted group algebra does not change under coordinate transformations of $\mathbb{F}_2^n$, the "if" part follows.

To prove the "only if" part of the theorem, we use Dickson's classification of quadratic forms on $\mathbb{F}_2^n$, and show that nonequivalent quadratic forms correspond to nonisomorphic algebras.

From Quadratic Forms to Algebras

Let us consider the quadratic forms $\mathbf{q}_0, \mathbf{q}_1$, and $\mathbf{q}_2$, and show that the corresponding twisted group algebras on $\mathbb{R}[\mathbb{F}_2^n]$ are precisely the matrix algebras appearing in the Chevalley classification of real Clifford algebras.

Consider first the 2-dimensional case, and the quadratic form $\mathbf{q}_0 = x_1 x_2$ on $\mathbb{F}_2^2$. The algebra $\mathbb{R}[\mathbb{F}_2^2]$ has two generators, e_1 and e_2. Since the polarization of $\mathbf{q}_0$ is the 2-form

$$\mathbf{b}(x, y) = x_1 y_2 + x_2 y_1,$$

the generators anticommute, and since $\mathbf{q}_0(\mathbf{x}_1) = \mathbf{q}_0(\mathbf{x}_2) = 0$, one has $e_1^2 = e_2^2 = \mathbf{1}$. This algebra is isomorphic to the algebra of real 2×2 matrices. Indeed, the following generators of $\mathrm{Mat}(2, \mathbb{R})$:

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

satisfy the above relations.

The quadratic form $\mathbf{q}_1 = x_1 x_2 + x_1 + x_2$ on $\mathbb{F}_2^2$ corresponds to the case $e_1^2 = e_2^2 = -\mathbf{1}$, since $\mathbf{q}_1(\mathbf{x}_1) = \mathbf{q}_1(\mathbf{x}_2) = 1$. Thus e_1 and e_2 generate the algebra of quaternions $\mathbb{H}$.

(1) Consider now the form $\mathbf{q}_0$ on $\mathbb{F}_2^{2k}$. The generators of the corresponding algebra structure on $\mathbb{R}[\mathbb{F}_2^{2k}]$ in each 2-dimensional block: $\{e_1, e_2\}, \{e_3, e_4\}, \ldots$ form a copy of $\mathrm{Mat}(2, \mathbb{R})$. The generators from different blocks commute, so that the algebra on $\mathbb{R}[\mathbb{F}_2^{2k}]$ is just the tensor product of k copies:

$$\mathrm{Mat}(2, \mathbb{R}) \otimes \mathrm{Mat}(2, \mathbb{R}) \otimes \cdots \otimes \mathrm{Mat}(2, \mathbb{R}) \simeq \mathrm{Mat}(2^k, \mathbb{R}).$$

Similarly, the algebra on $\mathbb{R}[\mathbb{F}_2^{2k}]$ corresponding to the form $\mathbf{q}_1$ is

$$\mathrm{Mat}(2, \mathbb{R})^{\odot(k-1)} \otimes \mathbb{H} \simeq \mathrm{Mat}(2^{k-1}, \mathbb{H}).$$

The forms $\mathbf{q}_0$ and $\mathbf{q}_1$ thus correspond to the algebras of real and quaternionic matrices, respectively.

(2) The form $\mathbf{q}_2$ on $\mathbb{F}_2^{2k+1}$ corresponds to the algebra

$$\mathrm{Mat}(2, \mathbb{R})^{\otimes k} \odot \mathbb{C} \simeq \mathrm{Mat}(2^k, \mathbb{C}),$$

since the last generator e_{2k+1} commutes with e_i for $i \leq 2k$ and squares to $-\mathbf{1}$, and therefore generates the algebra of complex numbers.

Finally, the forms $\mathbf{q}_0$ and $\mathbf{q}_1$ trivially extended to $\mathbb{F}_2^{2k+1}$ correspond to the algebras

$$\mathrm{Mat}(2^k, \mathbb{R}) \otimes \mathbb{R}^2 \quad \text{and} \quad \mathrm{Mat}(2^{k-1}, \mathbb{H}) \otimes \mathbb{R}^2,$$

respectively. These are just the double copies of the aforementioned matrix algebras.

We are ready to complete the proof of Theorem 1. Consider a twisted group algebra $(\mathbb{R}[\mathbb{F}_2^n], f)$ with bilinear function f, and the corresponding quadratic form α. By Dickson's theorem, every quadratic form on $\mathbb{F}_2^n$ of rank $2k$ is equivalent to $\mathbf{q}_0, \mathbf{q}_1$, or $\mathbf{q}_2$. We have just proved that $(\mathbb{R}[\mathbb{F}_2^n], f)$ must be isomorphic to a direct sum of several copies of $\mathrm{Mat}(2^k, \mathbb{R})$, $\mathrm{Mat}(2^{k-1}, \mathbb{H})$, or $\mathrm{Mat}(2^k, \mathbb{C})$, respectively. Since these three algebras are obviously not isomorphic to each other, we have proved that nonequivalent quadratic forms correspond to nonisomorphic algebras.

Comments

The above linear algebra considerations hide the cohomological nature of Theorem 1. A twisted group algebra is associative if and only if the function f satisfies

$$(\delta f)(x, y, z) := f(x, y) + f(x, y + z) + f(x + y, z) + f(y, z) = 0,$$

for all $x, y, z \in \mathbb{F}_2^n$. Such a function f is called a 2-cocycle on the abelian group $\mathbb{Z}_2^n$ (it is convenient to use this notation for $\mathbb{F}_2^n$ considered only as an abelian group, and not as a vector space). This condition is obviously satisfied if f is bilinear, but bilinearity is, of course, not necessary. The above condition is also always satisfied for the functions f of the form

$$f(x, y) = g(x + y) + g(x) + g(y),$$

where g is an arbitrary function of one argument. Then f is called a trivial cocycle, or a coboundary. For instance, the polar form $\mathbf{b}$ is the coboundary of $\mathbf{q}$. A statement closely related to Theorem 1 affirms that, for an arbitrary 2-cocycle f (not necessarily bilinear), the diagonal function $f(x, x)$ must be a quadratic form, and this form determines the cohomology class of f (for more details, see [12]).

A more general class of twisted group algebras $(\mathbb{R}[\mathbb{F}_2^n], f)$, which contains the classical algebra of octonions, was considered in [12]. These are algebras where f is not necessarily a 2-cocycle, but δf is a symmetric function of three arguments. In this case, $\alpha(x) := f(x, x)$ must be a cubic form on $\mathbb{F}_2^n$. Two such algebras are proved to be isomorphic (at least as $\mathbb{Z}_2^n$-graded algebras) if and only if the corresponding cubic forms are equivalent. Note that the classification of cubic forms on $\mathbb{F}_2^n$ is a difficult unsolved problem.

Quadratic forms over $\mathbb{F}_2$ have been recently used in [13] to classify gradings of simple real algebras by abelian groups. The quadratic forms $\mathbf{q}_0, \mathbf{q}_1$, and $\mathbf{q}_2$ appear explicitly in this classification. This result is of course closely related to real Clifford algebras; see in particular in Remark 17 of [13].

Clifford algebras were considered as superalgebras, that is, $\mathbb{Z}_2$-graded algebras, already in [2]. Thanks to Proposition 0.1, Clifford algebras with n generators can be understood as *graded-commutative* algebras over $\mathbb{Z}_2^n$ (also see [11] for a classification). This viewpoint was recently used to revisit the old classical problem of Cayley's, of developing linear algebra with coefficients in Clifford algebras, see [5] and references therein. In particular, it offers a new understanding of the Dieudonné determinant of quaternionic matrices.

ACKNOWLEDGMENTS

I am pleased to thank Alberto Elduque and Sophie Morier-Genoud for helpful comments. I am also grateful to the anonymous referee for for his/her comments and suggestions that allowed significant improvement and correction of this note.

Appendix 1: Classification of Quadratic Forms on $\mathbb{F}_2^n$

For the sake of completeness, let us give a proof of Dickson's theorem.

(1') Let $\mathbf{q}$ be a nondegenerate quadratic form on $\mathbb{F}_2^{2k}$. There exist coordinates on $\mathbb{F}_2^{2k}$ in which the polar bilinear form $\mathbf{b}$ associated to $\mathbf{q}$ is written as:

$$\mathbf{b}(x, y) = x_1y_2 + x_2y_1 + \cdots + x_{2k-1}y_{2k} + x_{2k}y_{2k-1}.$$

This is obvious since $\mathbf{b}$ is "skew symmetric" (alternating). Indeed, one choses the first coordinate axis in an arbitrary way, then the second axis such that $\mathbf{b}(\mathbf{x}_1, \mathbf{y}_2) \neq 0$, and the other coordinates are in the orthogonal complement.

The quadratic form $\mathbf{q}$ is then as follows:

$$\mathbf{q} = x_1x_2 + x_3x_4 + \cdots + x_{2k-1}x_{2k} + (\text{linear terms}). \quad (4)$$

Consider one of the binary terms x_ix_{i+1} of $\mathbf{q}$. Changing coordinates,

$$x_i' = x_i, \qquad x_{i+1}' = x_i + x_{i+1},$$

this term is equivalent to $x_ix_{i+1} + x_i$. Therefore, 2-dimensional blocks of $\mathbf{q}$ can be reduced to one of two types:

$$x_ix_{i+1} \quad \text{or} \quad x_ix_{i+1} + x_i + x_{i+1}.$$

Consider now a pair of blocks of the second type:

$$x_ix_{i+1} + x_jx_{j+1} + x_i + x_{i+1} + x_j + x_{j+1}.$$

The coordinate transformation:

$$x_i' = x_i + x_j, \quad x_{i+1}' = x_{i+1}, \quad x_j' = x_j, \quad x_{j+1}' = x_j + x_{i+1}.$$

sends this form to $x_ix_{i+1} + x_jx_{j+1}$.

It follows that $\mathbf{q}$ is equivalent to $\mathbf{q}_0$ if it contains an even number of blocks of the second type, or to $\mathbf{q}_1$, otherwise. As mentioned, $\mathbf{q}_0$ and $\mathbf{q}_1$ are not equivalent.

(2') Consider a form $\mathbf{q}$ of rank $2k$ on $\mathbb{F}_2^{2k+1}$. Choose coordinates in which the polar form $\mathbf{b}$ is as above, then $\mathbf{q}$ is again as in (4). If $\mathbf{q}$ contains the linear term x_{2k+1}, then the coordinate transformations $x_{2k+1}' = x_{2k+1} + x_i$ allow us to kill all other linear terms, so that $\mathbf{q}$ is equivalent to $\mathbf{q}_2$. Otherwise, the problem is reduced to Part (1').

Appendix 2: The Table of Clifford Algebras

The following well-known table of real Clifford algebras illustrates Chevalley's theorem,

$p \setminus q$	0	1	2	3	4	5
0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}^2$	$\mathrm{Mat}(2,\mathbb{H})$	$\mathrm{Mat}(4,\mathbb{C})$
1	$\mathbb{R}^2$	$\mathrm{Mat}(2,\mathbb{R})$	$\mathrm{Mat}(2,\mathbb{C})$	$\mathrm{Mat}(2,\mathbb{H})$	$\mathrm{Mat}(2,\mathbb{H})^2$	...
2	$\mathrm{Mat}(2,\mathbb{R})$	$\mathrm{Mat}(2,\mathbb{R})^2$	$\mathrm{Mat}(4,\mathbb{R})$	$\mathrm{Mat}(4,\mathbb{C})$	...	
3	$\mathrm{Mat}(2,\mathbb{C})$	$\mathrm{Mat}(4,\mathbb{R})$	$\mathrm{Mat}(4,\mathbb{R})^2$	...		
4	$\mathrm{Mat}(2,\mathbb{H})$	$\mathrm{Mat}(4,\mathbb{C})$	...			
5	$\mathrm{Mat}(2,\mathbb{H})^2$	...				

where the horizontal axis is parametrized by q and the vertical by p. The algebras with n generators are represented in the diagonal $p + q = n$.

The table can be filled using the first examples:

$$\mathrm{Cl}_{0,0} \simeq \mathbb{R}, \qquad \mathrm{Cl}_{1,0} \simeq \mathbb{R}^2, \qquad \mathrm{Cl}_{0,1} \simeq \mathbb{C}$$

and the following periodicity statements:

$$\mathrm{Cl}_{p,q+2} \simeq \mathrm{Cl}_{q,p} \otimes \mathbb{H}, \qquad \mathrm{Cl}_{p+1,q+1} \simeq \mathrm{Cl}_{p,q} \otimes \mathrm{Mat}(2,\mathbb{R}), \\ \mathrm{Cl}_{p+2,q} \simeq \mathrm{Cl}_{q,p} \otimes \mathrm{Mat}(2,\mathbb{R}),$$

where the tensor products are over $\mathbb{R}$. We will not prove them, but they can be obtained directly by comparing the generators of the algebras, see [10, 7], or by analyzing quadratic forms on $\mathbb{F}_2^n$. Let us mention that these periodicities imply the following remarkable periodicity modulo 8:

$$\mathrm{Cl}_{p,q+8} \simeq \mathrm{Cl}_{p+4,q+4} \simeq \mathrm{Cl}_{p+8,q} \simeq \mathrm{Cl}_{p,q} \otimes \mathrm{Mat}_{16}(\mathbb{R}),$$

called the Bott periodicity.

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